International Journal of Theoretical Physics, Vol. 21, Nos. 8/9, 1982

Scalar and Neutrino Fields in the Gödel Universe[1]

D. A. Leahy[2]

ES-62, Space Sciences Laboratory, NASA, Marshall Space Flight Center

Received September 17, 1981

Some properties of the Gödel universe are demonstrated, such as closed timelike lines, and new coordinates are found. The scalar and neutrino field equations are solved and the eigenvalue spectra are calculated. The scalar field has a discrete spectrum, but the neutrino field has, in addition, a continuous spectrum due to the coupling of neutrino spin and rotation in the Gödel universe. The mode solutions do not form a complete set for either the scalar or neutrino fields; therefore, a quantum field theory cannot be constructed in the usual manner.

1. INTRODUCTION

Conference 82 was held in honor of P. A. M. Dirac. His work lies at the foundation of much of physics, most notably quantum mechanics and, in particular, the Dirac equation. He has interests in the structure of the universe through his large numbers hypothesis. The following work is an application of field theory and quantum mechanics in the Gödel universe and is close to the research interests of Dirac.

The effects of the global properties of a space-time on quantized fields is a topic of current interest. More particularly, one would like to know whether (and how) the global structure of the universe affects local experiments. Global properties such as the expansion of the universe are observed through the red shift of distant objects; however, more truly local effects are of concern here. An example is the discreteness of the eigenvalue spectrum for a system of finite size (e.g., the standard undergraduate quantum mechanics problem of the particle in a box).

A simple space-time exhibiting a number of unusual properties is the Gödel universe. The Gödel universe exhibits properties associated with the

[1]Presented at the Dirac Symposium, Loyola University, New Orleans, May 1981.
[2]NAS/NRC Resident Research Associate.

0020-7748/82/0800-0703$03.00/0

rotation of the universe. It is homogeneous in space and time and is filled with a perfect fluid, just as the Friedman cosmologies. However, the presence of rotation has two major effects[3]: the space-time metric is not parity invariant, and closed timelike lines are present; ie., causality is violated. This latter property is commonly associated with rotation (e.g., Tipler, 1974). Some questions which arise are: How does the presence of closed timelike lines affect the propagation of a quantum field in spacetime? What effects does the lack of parity invariance have on the possible states for the parity noninvariant neutrino field? However, though the real universe undoubtedly possesses some small amount of rotation, closed timelike lines are unlikely to be present. Unfortunately, these two effects cannot be separated for a simple space-time such as the Gödel universe.

Section 2 is concerned with the symmetries of the Gödel metric, as expressed by the Killing vectors. These are used in deriving a new set of coordinates for the Gödel metric, with which a major part of the subsequent calculations are carried out. Using a variational method, geodesics for the Gödel metric are found in Section 3. The absence of closed timelike geodesics (i.e., absence of causality violation for an unaccelerated observer) is demonstrated, and in Section 4 the presence of closed timelike curves (with accelerations) is demonstrated with an example.

The scalar wave equation is solved in Section 5. This is done in the new coordinates and in Gödel's original coordinates. Explicit formulas for the eigenmodes and frequencies are given, and the frequency spectrum is plotted. In Section 6 the neutrino field is investigated, with similar results. The neutrino field is found to possess both continuous and discrete parts to its eigenvalue spectrum, whereas the scalar field has only discrete eigenvalues.

The problem of formulating a classical field theory is discussed in Section 7. In Section 8 the orthogonality and completeness of the mode solutions are discussed. Section 9 addresses the problem of quantizing the fields. Because of the absence of a complete Cauchy surface in the Gödel universe and the incompleteness of the mode solutions to the scalar and neutrino field equations over a three-dimensional surface, second quantization of the scalar and neutrino fields cannot be carried out in the usual manner. Section 10 summarizes the results obtained in this investigation.

Previous work includes that of Hiscock (1978), who has solved the scalar wave equation in the Gödel metric and has calculated the eigenvalue spectrum, and of Mashoon (1975), who has considered the propagation of electromagnetic waves and demonstrated the coupling between the helicity of the photon and the rotation of the Gödel universe.

[3]These and other properties are summarized in Gödel (1949).

2. GROUP STRUCTURE OF THE GÖDEL UNIVERSE

In this section, a new set of coordinates is derived for the Gödel metric. First, the symmetries of the metric are found by solving for the Killing vector fields of the Gödel metric. The commutation relations of the Killing vectors are then examined to find the structure coefficients of the symmetry group of the Gödel metric. Linear combinations of the Killing vectors are formed to simplify the commutation relations and group structure coefficients. A mutually commuting subset of these new Killing vectors is used to define a new set of coordinates for the Gödel metric.

We start by using the metric in Gödel's original coordinates (Ryan and Shepley, 1975):

$$\begin{aligned} ds^2 = & \left[dx^0 + \exp(x^1)\,dx^2\right]^2 - (dx^1)^2 \\ & - \exp(2x^1)(dx^2)^2/2 - (dx^3)^2 \end{aligned} \tag{1}$$

An isometry is a transformation which leaves a space-time metric invariant. A Killing vector field describes an infinitesimal isometry; i.e., the metric is left invariant by sliding the space-time along a Killing vector field. The Gödel metric possesses the five Killing vector fields:

$$\begin{aligned} &\eta_0^\mu = (1,0,0,0), \qquad \eta_1^\mu = (0,1,-x^2,0) \\ &\eta_2^\mu = (0,0,1,0), \qquad \eta_3^\mu = (0,0,0,1) \\ &\eta_4^\mu = \left(-2\exp(-x^1), x^2, \exp(-2x^1) - (x^2)^2/2, 0\right) \end{aligned} \tag{2}$$

These are found by solving Killing's equation:

$$L_\eta(g_{\mu\nu}) = \eta_{\mu;\nu} + \eta_{\nu;\mu} = 0 \tag{3}$$

where L_ρ is the Lie derivative with respect to the vector ρ. Equation (3) specifies that the derivatives of the metric functions $g_{\mu\nu}$ are zero in the direction of a Killing vector.

The Killing vectors of equations (2) satisfy the following commutation relations[4]:

$$[\eta_0, \eta_i] = [\eta_3, \eta_i] = 0, \qquad i = 0,1,2,3,4 \tag{4a}$$

$$[\eta_1, \eta_4] = \eta_4, \qquad [\eta_1, \eta_2] = \eta_2, \qquad [\eta_2, \eta_4] = \eta_1 \tag{4b}$$

[4]For any two vector fields ρ, σ the Lie derivative of σ with respect to ρ is given by $L_\rho(\sigma) = [\rho, \sigma]$.

If one defines new Killing vectors[5] by

$$\tilde{\eta}_2 = (\eta_2 + \eta_4)/\sqrt{2}, \qquad \tilde{\eta}_4 = (\eta_2 - \eta_4)/\sqrt{2}, \qquad \tilde{\eta}_1 = \eta_1 \tag{5}$$

then the commutation relations (4b) of the subspace spanned by $\tilde{\eta}_1$, $\tilde{\eta}_2$, and $\tilde{\eta}_4$ have the form

$$[\tilde{\eta}_i, \tilde{\eta}_j] = C^k_{ij}\tilde{\eta}_k \tag{6}$$

The C^k_{ij} are the structure constants for the symmetry group of the Gödel metric. They have the form

$$C^4_{12} = C^2_{14} = C^1_{42} = 1, \qquad C^k_{ij} = -C^k_{ji} \tag{7}$$

others zero.

These are the same as the structure constants for the Lorentz group of space-time rotations (i.e., Lorentz boosts and ordinary rotations) in a (2+1)-dimensional Minkowski space. This is not surprising since the isotropy group of a point must be a subgroup of the homogeneous Lorentz group (Ryan and Shepley, 1975). The isotropy group of a point P is the set of isometries which leave P fixed. Transformations which have a fixed point are rotations. The only Killing vector of the set (2) which has a fixed point is η_4, and only at $x^2 = 0$, x^1 approaches infinity. None of the new Killing vectors $\tilde{\eta}_i$ [equation (5)] has a fixed point. However, one can construct rotations with finite fixed points by simple linear combinations of the Killing vectors (2). For example, $\eta_4 + 2\eta_0 - \eta_1 + 3\eta_2/2$ vanishes on the 2-plane $x^1 = 0$, $x^2 = 1$. Killing vectors describe only infinitesimal isometries. Thus, there is no reason why the path of a point undergoing a macroscopic rotation should close to form a circle. In general, a rotational isometry will take a point along a corkscrewlike path. The five Killing vectors of equation (2) express the entire set of symmetries of the Gödel metric, including homogeneity in space and time. Only three of these are simple translations, in the coordinates of equation (1).

According to equation (4), only three of the five Killing vectors mutually commute. By choosing three mutually commuting Killing vector fields to form coordinate lines, one can obtain metric coefficients which depend functionally only on the fourth coordinate. The first three coordinates correspond to symmetries of the Gödel universe. η_0, η_1, η_3 are chosen here as the three Killing vectors with which to construct new coordinates.

[5]In general, linear combinations of Killing vectors are also Killing vectors only if the coefficients are constants.

They then have the following form in the new coordinates:

$$\eta_0^\mu = (1,0,0,0), \qquad \eta_1^\mu = (0,1,0,0), \qquad \eta_3^\mu = (0,0,0,1) \tag{8}$$

The new coordinates are labeled by

$$x^\mu = (t, \phi, r, x) \tag{9}$$

so that η_1 corresponds to translation along the new coordinate ϕ, for example.

To find the coordinate transformation of new from old coordinates we take

$$x^3 = x \tag{10}$$

since they correspond to the same Killing vector [by equations (8) and (2)]. Any vector ρ has components in two coordinate systems, x^μ and $\bar{x}^\mu$, related by the transformation

$$\bar{\rho}^\mu = (d\bar{x}^\mu/dx^\nu)\rho^\nu \tag{11}$$

With the known forms of η_0, η_1 and $\bar{\eta}_0$, $\bar{\eta}_1$, one obtains, from equation (11),

$$1 = dx^0/dt, \qquad 0 = dx^1/dt, \qquad 0 = dx^2/dt \tag{12a}$$

$$0 = dx^0/d\phi, \qquad 1 = dx^1/d\phi, \qquad -x^2 = dx^2/d\phi \tag{12b}$$

This allows one to write the coordinate transformation as

$$x^0 = t + M(r), \qquad x^1 = \phi + h(r), \qquad x^2 = r\exp[-\phi - h(r)] \tag{13}$$

with $M(r)$ and $h(r)$ arbitrary functions. The x^2 dependence on r has been chosen to be $r\exp(-x^1)$ for simplicity. Since the r coordinate can still undergo a scale change, it forms the arbitrary function for x^2 in equation (13). One can specify $M(r)$ and $h(r)$ by imposing conditions on the form of the metric in the new coordinates. The metric is, from equations (1) and (13)

$$\begin{aligned} ds^2 = {} & \{dt + M'dr + [-r d\phi + (1 - rh')dr]\}^2 \\ & - (d\phi + h'dr)^2 - r^2[d\phi + (h' - 1/r)dr]^2/2 - (dx)^2 \end{aligned} \tag{14}$$

with $' = d/dr$. Choosing $M' = rh' - 1$ eliminates the cross term in $dt\,dr$. Furthermore, setting $h' = r/(2 + r^2)$ eliminates the $d\phi\,dr$ cross term. With these simplifications the metric is

$$ds^2 = (dt - r d\phi)^2 - (2+r^2)(d\phi)^2/2 - (dr)^2/(2+r^2) - (dx)^2 \quad (15)$$

The metric (15) can be put in a more natural form by transforming the r coordinate to θ by a scale change:

$$r = \sqrt{2}\sinh\theta \quad (16)$$

This yields the final form of the new metric[6] which we will use:

$$ds^2 = \left(dt - \sqrt{2}\sinh\theta\, d\phi\right)^2 - \cosh^2\theta (d\phi)^2 - (d\theta)^2 - (dx)^2 \quad (17)$$

The coordinate transformation is, in summary

$$x^0 = t - \tanh\theta, \qquad x^1 = \phi - 1/\left(\sqrt{2}\cosh\theta\right) \quad (18a)$$

$$x^2 = \sqrt{2}\sinh\theta \exp(-x^1), \qquad x^3 = x \quad (18b)$$

The two Killing vectors other than those specified in equation (8) [i.e., η_2 and η_4 of equation (2)] are

$$\eta_2^\mu = \exp(x^1)\left[1/\left(\sqrt{2}\cosh^3\theta\right), -\sinh\theta/(2\cosh^3\theta), 1/\left(\sqrt{2}\cosh\theta\right), 0\right]$$

$$\eta_4^\mu = \exp(-x^1)\left[1/\left(\sqrt{2}\cosh\theta\right), \sinh\theta\left(\sqrt{2} - 1/(2\cosh\theta)\right), \cosh\theta/\sqrt{2}, 0\right] \quad (19)$$

in the new coordinates t, ϕ, θ, and x. The preceding x^1 is understood to be a function of ϕ and θ, as given in equation (18).

3. GEODESICS

In this section we study the geodesics of the Gödel universe. A geodesic is a path (either spacelike, timelike, or null) for which the tangent vector U to the path obeys $U^\alpha_{;\beta}U^\beta = aU^\alpha$. $U^\alpha_{;\beta}U^\beta$ is the derivative of U in the direction of U. Thus, along the path of a geodesic the tangent vector is transported parallel to itself. The path is specified by four differentiable functions of some parameter. By transforming to a new parameter, one can

[6] This form was pointed out to me by W. G. Unruh.

always put the geodesic equation in the form $U^{\alpha}_{;\beta}U^{\beta}=0$. Such a parameter is referred to as an affine parameter. For timelike (or spacelike) geodesics, an affine parameter is proportional to proper time (or distance). Regardless of parametrization, timelike and null geodesics correspond to the force-free motion of massive and massless particles, respectively.

A segment of a geodesic (in general, entire geodesics have no end-points) is an extremal path between its end points. All geodesic segments, and thus all geodesics, can be found by solving the variational problem:

$$0=\delta\int L\,ds \tag{20}$$

with s an affine parameter (proper time or distance for timelike or spacelike geodesics). The Langrangian L is proportional to the square of the proper distance along the curve as a function of the parameter s:

$$L=g_{\mu\nu}(dx^{\mu}/ds)(dx^{\nu}/ds)$$

Extremizing this can be shown to be equivalent to extremizing the proper distance. For the Gödel metric of equation (17), L is given by

$$L=\left(dt/ds-\sqrt{2}\sinh\theta\,d\phi/ds\right)^2-\cosh^2\theta(d\phi/ds)^2 -(d\theta/ds)^2-(dx/dx)^2 \tag{21}$$

Lagrange's equations for t, ϕ, and x (with $\cdot=d/ds$) yield the constants of the motion:

$$A=\dot{t}-\sqrt{2}\sinh\theta\dot{\phi},\qquad B=\dot{x},\qquad C=\sqrt{2}\sinh\theta A+\cosh^2\theta\dot{\phi} \tag{22}$$

Substituting into equation (17) yields (for timelike geodesics) an equation for $\theta(s)$:

$$1=A^2-\dot{\theta}^2-\left(C-\sqrt{2}\sinh\theta A\right)^2/\cosh^2\theta-B^2 \tag{23}$$

If one defines the constants D and E and the function y by

$$D=A^2+B^2+1,\qquad E=2A^2C^2/D+A^2-B^2-C^2-1 \tag{24a}$$

$$y=\sinh\theta-\sqrt{2}AC/D \tag{24}$$

then one can rewrite (23) as

$$\dot{y}^2 + Dy^2 = E \tag{25}$$

For null geodesics the left-hand side of equation (23) is zero. This results in equations (24) and (25) being valid but with D and E both defined without the "1" in them. Differentiation of equation (3.6) with respect to s yields

$$2\dot{y}(\ddot{y} + Dy) = 0 \tag{26}$$

Thus, y is either a constant along a geodesic ($y = E/D$) or y obeys the simple harmonic oscillator equation[7]:

$$y = (E/D)^{1/2}\sin(D^{1/2}s + \alpha) \tag{27}$$

From equation (22), t and ϕ satisfy the first-order equations:

$$\dot{t} = -A + \left[2A(1 + C^2/D) + \sqrt{2}\,Cy\right)/\left(y^2 + 2\sqrt{2}\,ACy/D + 1 + 2A^2C^2/D^2\right] \tag{28a}$$

$$\dot{\phi} = \left[C(1 - 2A^2/D) - \sqrt{2}\,Ay\right)/\left(y^2 + 2\sqrt{2}\,ACy/D + 1 + 2A^2C^2/D^2\right] \tag{28b}$$

A, B, and C are arbitrary subject to $E > 0$ by equation (25). The explicit form of any geodesic is given by equations (24), (21), and a direct integration[8] of equation (28) with respect to s. In the case in which y is constant, $\dot{t}$ and $\dot{\phi}$ are constants.

We now demonstrate the absence of closed timelike geodesics[9] in the maximal manifold. For constant y, we could regard t or ϕ as periodic coordinates to obtain closed timelike geodesics. However, to obtain a maximal manifold, any such periodicities in the coordinates have been unwrapped.[10] From the preceding equation (28), for periodic y one has that θ, $\dot{t}$, and $\dot{\phi}$ are periodic. However, t cannot be periodic unless A is zero by equation (28). Equation (24) then gives a negative value for E, which is not allowable. Since the preceding variational approach gives all the geodesics, this implies that there are no closed timelike geodesics.

[7]Consider negative E: no solution to equation (25) exists. Thus, values of A, B, and C which result in negative E are not allowed.

[8]The result is rather complicated and is not given here.

[9]Chandrasekhar and Wright (1961) have shown that no closed timelike geodesics exist.

[10]Hiscock (1978) uses a periodic coordinate ϕ to give an example of a closed timelike curve.

4. CLOSED TIMELIKE LINES

The existence of closed timelike lines is examined in this section. The x coordinate can be ignored, so we consider only the three-space with coordinates t, θ, ϕ and metric

$$ds^2 = \left(dt - \sqrt{2}\sinh\theta\, d\phi\right)^2 - \cosh\theta^2 (d\phi)^2 - (d\theta)^2 \tag{29}$$

Ryan and Shepley (1975) give an explicit form for closed timelike curves in the coordinates of Gödel, i.e., those of equation (1):

$$x^0 = A[\sin(z) - \sin(z)\cos(z)/2], \qquad x^1 = -B\cos(z)$$
$$x^2 = -A\sin(z), \qquad x^3 = 0 \tag{30}$$

The constants A and B need to be chosen properly and z is unbounded. In the coordinates here, t, ϕ, θ, and x, this takes the form

$$t = A\sin(z)[1 - \cos(z)/2]$$
$$- \{1 + 2/[A\sin(z)\exp(-B\cos(z))]\}^{-1/2}$$
$$\phi = -B\cos(z) + [A^2\sin^2(z)\exp[-2B\cos(z)] + 2]^{-1/2}$$
$$\sinh\theta = -A\sin(z)\exp[-B\cos(z)]/2, \qquad x = 0 \tag{31}$$

The nature of the light cones (for which $ds^2 = 0$) is examined to illustrate the nature of the closed timelike lines.[11] Write $\cdot = d/d\lambda$ for some suitable affine parameter λ. Then one has

$$0 = \dot{t}^2 - 2\sqrt{2}\sinh\theta\,\dot{t}\dot{\phi} + (\sinh^2\theta - 1)\dot{\phi}^2 - \dot{\theta}^2 \tag{32}$$

The infinitesimal light cones will be described by the surface defined by equation (32) in the tangent space. The only variable in equation (32) is θ, so one can illustrate the behavior of these surfaces for various θ. This is done in Figure 1. The light cones open up as one moves away from $\theta = 0$, and rotate counterclockwise for increasing θ, clockwise for decreasing θ. The positive t direction is always inside the forward light cone, and the negative t direction is always inside the backward light cone. At $\sinh\theta = 1$, the positive

[11]Hawking and Ellis (1973) use another set of coordinates to illustrate the light cones, null geodesics, and an example of a closed timelike curve.

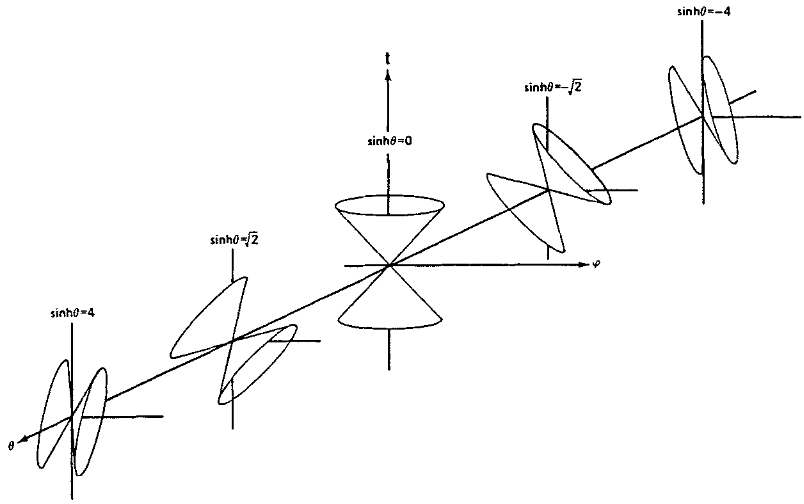

Fig. 1. Light cones in the Gödel universe.

ϕ direction is tangent to the backward light cone, and the negative ϕ direction is tangent to the forward light cone. For $\sinh\theta = -1$ the situation is reversed. For $|\sinh\theta| > 1$, the ϕ direction is inside the double light cone.

With the light-cone structure established, one can construct a closed timelike curve as follows, for example (this is illustrated by Figure 2):

(1) Move along positive θ and t in the $\theta - t$ plane past $\sinh\theta = 1$ where the edge of the light cone dips below the constant t plane (say, until $\sinh\theta = 4$).

(2) Change direction to negative t (still inside the forward light cone) and move in the $t - \phi$ plane at $\sinh\theta = 4$.

(3) At negative ϕ, large negative t change direction to be in the $\theta - \phi$ plane and move along negative θ and ϕ until almost at $\sinh\theta = 1$ (for $|\sinh\theta| < 1$, the forward light cone no longer tips below the constant t plane).

(4) Move along positive t and ϕ, negative θ back to the origin.

The tangent vector to the previously described path has remained inside the forward light cone for the entire closed path, and thus the path is a valid closed timelike curve. One can smooth out the corners to get reasonable accelerations. However, as we have shown, no closed timelike lines exist which are also geodesics.

As opposed to the case for geodesics, there are no general methods which give all the closed timelike lines. They must be found by trial and error. Because of this, we have only given an example of one closed timelike

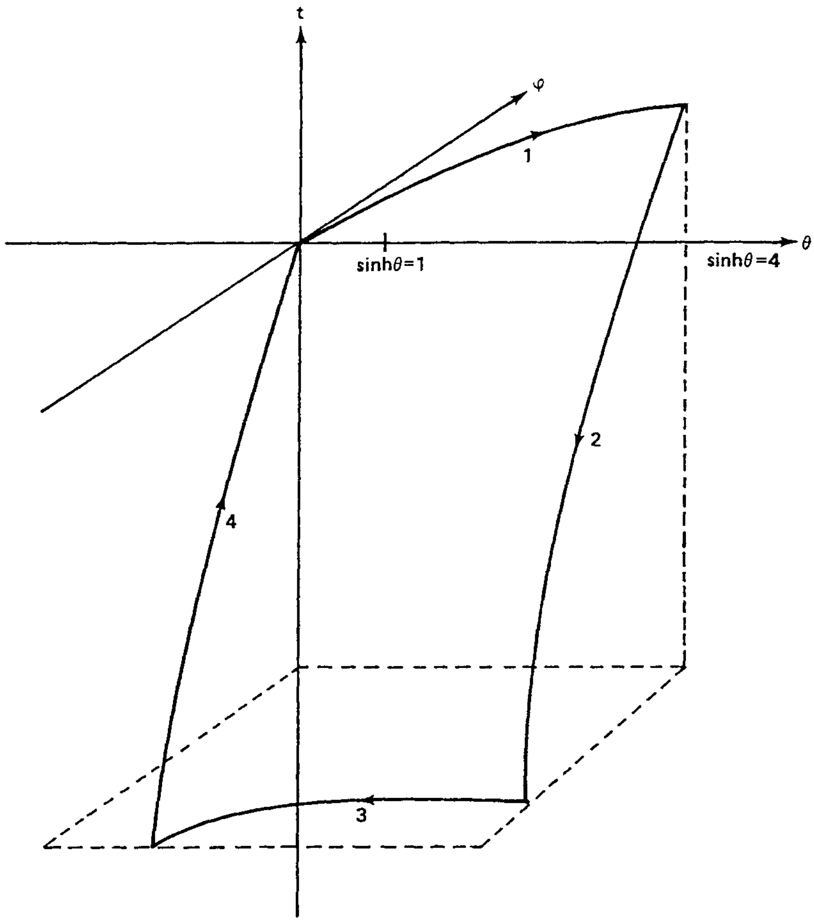

Fig. 2. An example of a closed timelike curve in the Gödel universe.

line (clearly, though, there are an infinite number of similar characters to the one illustrated in Figure 2). We leave the topic of paths of classical point particles and, in the remaining sections, study the behavior of fields in the Gödel universe, starting with the scalar field.

5. SCALAR FIELD

In this section, solutions for the scalar field equation in the Gödel metric are found. This is done in the coordinates derived in Section 2 and in

Gödel's original coordinates of equation (1). In Section 7 the question of constructing a classical field theory is addressed. Since the Gödel metric possesses no complete spacelike surfaces, which are desired as initial data surfaces for the fields, one is faced with making an arbitrary choice of constant "time" surfaces. Use of two different choices, i.e., two coordinate systems, helps illustrate the effect of this arbitrariness.

For the purposes of finding mode solutions and eigenfrequencies, one set of coordinates suffices. Fortuitously, in Gödel coordinates, the number of parameters involved can be reduced by one via a mode-dependent transformation. As a result, the solutions in this case yield an exact formula for the mode frequencies. However, the features of the mode functions and frequency spectrum are the same for both sets of coordinates.

The minimally coupled massless scalar field $\tilde{\phi}$ satisfies the covariant Klein–Gordon equation:

$$(\tilde{\phi}_{,\nu} g^{\mu\nu})_{;\nu} = 0 \quad \text{or} \quad \left(1/\sqrt{-g}\right)\left(\sqrt{-g}\, g^{\mu\nu}\tilde{\phi}_{,\mu}\right)_{,\nu} = 0 \tag{33}$$

where the semicolon means covariant derivative and the comma means partial derivative. g is the determinant of the matrix of metric coefficients $g_{\mu\nu}$. In the coordinates of equation (17), this becomes

$$\begin{aligned}(1-2\tanh^2\theta)\tilde{\phi}_{,00} - 2\sqrt{2}\,(\sinh\theta/\cosh^2\theta)\tilde{\phi}_{,01} \\ -(1/\cosh^2\theta)\tilde{\phi}_{,11} - \tilde{\phi}_{,22} - \tanh\theta\tilde{\phi}_{,2} - \tilde{\phi}_{,33} = 0\end{aligned} \tag{34}$$

g is $-\cosh^2\theta$ in these coordinates. One can define simultaneous eigenmodes by exploiting the isometries of the Gödel metric through the Killing vectors η_i. This is achieved by imposing periodicity of the modes along directions given by a set of commuting Killing vector fields:

$$L_{\eta_i}\tilde{\phi} = -i\alpha_i\tilde{\phi}, \qquad i = 0, 1, 3 \tag{35}$$

Here L_ρ stands for the Lie derivative with respect to the vector ρ. The α_i are the momenta of the field $\tilde{\phi}$ in the directions η_i. In equation (35) we have chosen the same Killing vectors and thus the same symmetries of the Gödel metric that were used in Section 2. Since three of the coordinates correspond directly to the Killing vectors used in equation (35), the Lie derivatives reduce to partial derivatives. One is then led directly to write the separated form of the scalar field, equation (37).

Separation of variables in equation (34) yields the following equation in θ:

$$\left[(2\tanh^2\theta-1)\alpha_0^2+2\sqrt{2}\,(\sinh\theta/\cosh^2\theta)\alpha_0\alpha_1+(1/\cosh^2\theta)\alpha_1^2+\alpha_3^2\right]$$
$$\times\tilde{\phi}-\tilde{\phi}_{,22}-\tanh\theta\,\tilde{\phi}_{,2}=0 \tag{36}$$

with

$$\tilde{\phi}(\alpha_i;x^\mu)=\exp(-i\alpha_0 t-i\alpha_1\phi-i\alpha_3 x)\tilde{\phi}(\alpha_i;\theta) \tag{37}$$

In terms of the variable $y=\sinh\theta$, this can be rewritten as

$$d/dy\left[(1+y^2)\,d\tilde{\phi}/dy\right]$$
$$+\left[-(\alpha_0^2+\alpha_3^2)+\left(2\alpha_0^2-\alpha_1^2-2\sqrt{2}\,\alpha_0\alpha_1 y\right)/(1+y^2)\right]\tilde{\phi}=0 \tag{38}$$

Examination of the singular points of this equation reveals that it is one form of the hypergeometric equation. Equation (38) can be rewritten in the form

$$\tilde{\phi}''+2y/(1+y^2)\tilde{\phi}'+\left[-k_1/(1+y^2)+(k_2-k_3y)/(1+y^2)^2\right]\tilde{\phi}=0 \tag{39}$$

with $'=d/dy$ and

$$k_1=\alpha_0^2+\alpha_3^2,\qquad k_2=2\alpha_0^2-\alpha_1^2,\qquad k_3=2\sqrt{2}\,\alpha_0\alpha_1 \tag{40}$$

The solution (Morse and Feschbach, 1953) of equation (39) can be written in terms of the Riemann symbol as

$$\tilde{\phi}=P\begin{Bmatrix} i & -i & \infty & y \\ \lambda & \mu & \nu & \\ \lambda' & \mu' & \nu' & \end{Bmatrix} \tag{41}$$

with $\lambda,\lambda';\mu,\mu';\nu,\nu'$ being the indices at the regular singular points at i, $-i$, and infinity, respectively. These pairs are given by the roots of

$$\lambda^2=(k_2-k_3 i)/4,\qquad \mu^2=(k_2+k_3 i)/4,\qquad \nu^2=k_1 \tag{42}$$

One can express $\tilde{\phi}$ in terms of the hypergeometric function, which is the analytic solution about $z=0$ of the hypergeometric equation:

$$z(z-1)F''(z)+[(a+b+1)z-c]F'(z)+abF(z)=0 \tag{43}$$

with $'=d/dx$. F can also be written in terms of the Riemann symbol:

$$F(a,b,c;z)=P\begin{Bmatrix} 0 & 1 & \infty & z \\ 0 & 0 & a & \\ 1-c & c-a-b & b & \end{Bmatrix} \tag{44}$$

To put the equation for $\tilde{\phi}$, equation (39), in the form of that for F, equation (43), both the dependent and independent variable must be transformed. Applying the required transformations, one finds $\tilde{\phi}$ in terms of F:

$$\begin{aligned}\tilde{\phi}=&AF(\lambda+\mu+\nu,1-\nu-\lambda'-\mu',\lambda-\lambda'+1;(1+iy)/2)\\ &+BF(\lambda'+\mu+\nu,1-\nu-\lambda-\mu',\lambda'-\lambda+1;(1+iy)/2)\end{aligned} \tag{45}$$

with A and B arbitrary constants. This is the general solution of equation (39) about the singular point $y=i$. The general solution about the singular point $y=-i$ is

$$\begin{aligned}\tilde{\phi}=&AF(\mu+\lambda+\nu,1-\nu-\mu'-\lambda',\mu-\mu'+1;(1-iy)/2)\\ &+BF(\mu'+\lambda+\nu,1-\nu-\mu-\lambda',\mu'-\mu+1;(1-iy)/2)\end{aligned} \tag{46}$$

The radius of convergence of both these solutions is 2. The solution about the point at infinity is

$$\begin{aligned}\tilde{\phi}=&A(y+i)^{\mu}/(y-i)^{\mu+\nu}F(\nu+\mu+\lambda,1-\lambda-\nu'-\mu',\\ &\nu-\nu'+1;2/(1+iy))+B(y+i)^{\mu}/(y-i)^{\mu+\nu}\\ &\times F(\nu'+\mu+\lambda,1-\lambda-\nu-\mu',\nu'-\nu+1;2/(1+iy))\end{aligned} \tag{47}$$

From equation (42), $\nu'=-\nu$ is real since $k_1>0$. Thus, one must have either A or B zero in equation (47) (depending on whether one chooses ν or ν' to be negative) so that $\tilde{\phi}$ does not diverge. Together, the equations (43), (46), and (47) allow one to find $\tilde{\phi}$ anywhere in the complex θ plane, in particular for real θ.

The general self-adjoint differential equation for the eigenvalue problem[12] over an interval of y is

$$[p(y)\tilde{\phi}'(y)]' + [q(y) + \lambda r(y)]\tilde{\phi}(y) = 0 \tag{48}$$

with $' = d/dy$ and $p(y)r(y)$ positive over the interval. The eigenvalue problem consists of finding all functions $\tilde{\phi}_n(y)$ which solve equation (48) and satisfy the boundary conditions, and of finding the eigenvalues λ_n that λ must take for each of the functions $\tilde{\phi}_n$. It can be shown that there are an infinite number of eigenvalues, ranging from some minimum value to positive infinity (Morse and Feschbach, 1953, Chapter 6). One can compare equation (38) or (39) with equation (48). One sees that either $-k_1$ or $+k_2$ can be regarded as the eigenvalue λ. The remaining k_i act as parameters:

$$\lambda = -k_1, \qquad r = 1, \qquad k_2, k_3 \text{ parameters} \tag{48a}$$

$$\lambda = +k_2, \qquad r = 1/(1+y^2), \qquad k_1, k_3 \text{ parameters} \tag{48b}$$

The question of orthogonality and completeness of the eigenfunctions is considered here. First we consider only the θ dependence of the mode solutions. Orthogonality and completeness of the eigenfunctions must be discussed in terms of an inner product. For the self-adjoint differential equation (48) the inner product of two solutions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ is normally defined as

$$(\tilde{\phi}_1, \tilde{\phi}_2) = \int dy\, \tilde{\phi}_1^*(y)\tilde{\phi}_2(y)r(y) \tag{49a}$$

Orthogonality of $\tilde{\phi}_1$ and $\tilde{\phi}_2$ is expressed by $(\tilde{\phi}_1, \tilde{\phi}_2) = 0$. Completeness of the eigenfunction set means that the difference between an arbitrary function $f(y)$ and its expansion $g_m(y)$ in terms of eigenfunctions $\tilde{\phi}_n(y)$

$$g_m(y) = \sum_{n=1}^{m} C_n\tilde{\phi}_n(y)$$

with C_n constants, can be made arbitrarily small; i.e., $(f - g_m, f - g_m)$ approaches 0 as m approaches infinity. For equation (48), completeness

[12]Suitable boundary conditions must be imposed at the end points of the interval.

can be expressed by the relation

$$\sum_{n=1}^{\infty} \tilde{\phi}_n(y)\tilde{\phi}_n^*(y')r(y) = \delta(y-y') \tag{49b}$$

where the sum is over all eigenfunctions.

The eigenfunctions for cases (48a) or (48b) would form a complete set if the eigenvalue λ were free to assume arbitrary values. However, from equation (40) one has $-k_1 = -(\alpha_0^2 + \alpha_3^2)$. Thus, k_1 cannot take on all eigenvalues required. In fact, since the eigenvalues of equation (48) range from some minimum value to positive infinity, $-k_1$ takes on only a finite subset of the infinite range of eigenvalues. k_2 can yield all eigenvalues above $-\alpha_1^2$ by varying[13] α_0. If the minimum eigenvalue for given α_1 and α_3 is less than $-\alpha_1^2, k_2$ cannot take on the first few eigenvalues. However, both choices (48a) and (48b) must lead to the same results for the full set of solutions for $\tilde{\phi}$ and the associated allowable values for α_0, α_1, and α_3. Later we will find it advantageous to use the choice equation (48a).

We first consider the case (48b) with k_2 as eigenvalue. For the self-adjoint differential equation (39), the eigenfunctions can be shown to be orthogonal (suitably normalized):

$$\int dy\, \tilde{\phi}(\alpha_1, \alpha_3, k_2, y)\tilde{\phi}^*(\alpha_1, \alpha_3, k_2', y)/(1+y^2) = \delta_{k_2, k_2'} \tag{50a}$$

or in terms of θ:

$$\int d\theta\, \tilde{\phi}(\alpha_1, \alpha_3, k_2, \theta)\tilde{\phi}^*(\alpha_1, \alpha_3, k_2', \theta)/\cosh\theta = \delta_{k_2, k_2'} \tag{50b}$$

If, in addition, k_2 takes on all possible eigenvalues in the spectrum of the self-adjoint differential operator (with boundary conditions), one has the completeness relation:

$$\sum_{k_2} \tilde{\phi}(\alpha_1, \alpha_3, k_2, y)\tilde{\phi}^*(\alpha_1, \alpha_3, k_2, y')/(1+y^2) = \delta(y-y') \tag{51a}$$

or in terms of θ:

$$\sum_{k_2} \tilde{\phi}(\alpha_1, \alpha_3, k_2, \theta)\tilde{\phi}^*(\alpha_1, \alpha_3, k_2, \theta')/\cosh\theta = \delta(\theta-\theta') \tag{51b}$$

[13] Take α_1 and α_3 as free parameters. Then α_0 is determined by the required eigenvalues of k_2.

For the choice that $-k_1$ is eigenvalue, equation (48a), the orthogonality relation is

$$\int dy\,\tilde{\phi}(\alpha_1,\alpha_3,-k_1,y)\tilde{\phi}^*(\alpha_1,\alpha_3,-k_1',y)=\delta_{k_2,k_1'} \tag{52a}$$

In terms of θ this becomes:

$$\int d\theta\,\tilde{\phi}(\alpha_1,\alpha_3,-k_1,\theta)\tilde{\phi}^*(\alpha_1,\alpha_3,-k_1,\theta)\cosh\theta=\delta_{k_1,k_1'} \tag{52b}$$

If $-k_1$ could take on all eigenvalues, then the completeness relation would be

$$\sum_{k_1}\tilde{\phi}(\alpha_1,\alpha_3,-k_1,y)\tilde{\phi}^*(\alpha_1,\alpha_3,-k_1,y')=\delta(y-y') \tag{53a}$$

or in terms of θ

$$\sum_{k_1}\tilde{\phi}(\alpha_1,\alpha_3,-k_1,\theta)\tilde{\phi}^*(\alpha_1,\alpha_3,-k_1,\theta')\cosh\theta=\delta(\theta-\theta') \tag{53b}$$

To determine the character of the eigenvalue spectrum, consider the form of equation (36) as θ approaches infinity:

$$d^2\tilde{\phi}/d\theta^2 \pm d\tilde{\phi}/d\theta - k_1\tilde{\phi}=0, \qquad \theta\to\pm\infty \tag{54}$$

This has asymptotic solutions:

$$\begin{aligned}\tilde{\phi}&=A\exp[-(\alpha+1/2)\theta]+B\exp[(\alpha-1/2)\theta], &&\theta\to+\infty\\ \tilde{\phi}&=C\exp[-(\alpha-1/2)\theta]+D\exp[(\alpha+1/2)\theta], &&\theta\to-\infty\end{aligned} \tag{55}$$

with $\alpha=+(1/4+k_1)^{1/2}$. In equations (55) C and D are functions of the arbitrary constants A and B, found by solving the full equation (39). When the boundary conditions are applied, one requires B and C to be zero [see also equation (47)] since one has $\alpha\geqslant 1/2$ for all values of α_0 and α_1. This will occur for only isolated values of the constants $-k_1$, k_2, and k_3 so that the eigenvalue spectrum is discrete.

It turns out that the mode solutions and eigenvalue spectrum are expressed more neatly by using the original coordinates of Gödel, equation

(1).[14] Substituting for $g^{\mu\nu}$ and g [$= -\exp(2x^1)/2$], one obtains the partial differential equation

$$\tilde{\phi}_{,00} + \tilde{\phi}_{,11} + \tilde{\phi}_{,1} + \tilde{\phi}_{,33} - 4\exp(-x^1)\tilde{\phi}_{,02} + 2\exp(-2x^1)\tilde{\phi}_{,22} = 0 \quad (56)$$

To separate variables in equation (56) we write $\tilde{\phi}$ as

$$\tilde{\phi} = f(x^1)\exp\left[i\left(k_2 x^2 + k_3 x^3 - wx^0\right)\right] \quad (57)$$

Then f satisfies

$$d^2f/(dx^1)^2 + df/dx^1 - \left(w^2 + k_3^2 + 4\exp(-x^1)wk_2 + 2k_2^2\exp(-2x^1)\right)f = 0 \quad (58)$$

For $k_2 = 0$ we can write f as

$$f(x^1) = A\exp(px^1) \quad (59)$$

Then p must satisfy

$$p^2 + p - \left(w^2 + k_3^2\right) = 0 \quad \text{or} \quad p = -(1/2) \pm \left(w^2 + k_3^2 + 1/4\right)^{1/2}$$

For f to converge for $-\infty < x^1 < +\infty$, p must vanish, i.e.,

$$w = k_3 = 0 \quad (60)$$

when $k_2 = 0$.

For k_2 nonzero define the variable z by

$$z = 2\sqrt{2}\,|k_2|\exp(-x^1) \quad (61)$$

This is not a transformation to new coordinates since k_2 depends on the mode under consideration [see equation (57)]. In terms of z the equation (58) for f becomes

$$d^2f/dz^2 + \left[-1/4 - \sqrt{2}\,mw/z - \left(w^2 + k_3^2\right)/z^2\right]f = 0 \quad (62)$$

with m defined by $m = k_2/|k_2| = \pm 1$. By performing the transformation of

[14]The solution of the scalar field equation in Gödel's coordinates has also been considered by Hiscock (1978), with similar results.

variables we have reduced the number of parameters from 3 to 2 (k_2 no longer appears). The equation (62) has a regular singularity at $z=0$ and an irregular singularity at infinity. We put equation (62) in the form given in the Appendix by equation (A.1). The values of k, α, λ, and λ' are then given by

$$k=1/2, \qquad \alpha=-mw/\sqrt{2}$$

$$\lambda=1/2+\left(w^2+k_3^2+1/4\right)^{1/2}, \qquad \lambda'=1/2-\left(w^2+k_3^2+1/4\right)^{1/2} \tag{63}$$

Since we have $\lambda' \leqslant 0$, convergent solutions are (see Appendix)

$$f(z)=A\exp(-kz)z^{\lambda}F(a,c,2kz) \tag{64}$$

with

$$c=1+\lambda-\lambda'=1+2\left(w^2+k_3^2+1/4\right)^{1/2}$$

$$a=c/2-\alpha/k=c/2+\sqrt{2}\,mw \tag{65}$$

where $a=-n$, $n=0,1,2,\ldots$ When a is a negative integer, the confluent hypergeometric function F reduces to the generalized Laguerre polynomials $L_n^{(c-1)}(2kz)$ (Abramnowitz and Stegun, 1964)[15]:

$$F(-n,c,2kz)=n!L_n^{(c-1)}(2kz)/(c)_n \tag{66}$$

The eigenfrequencies are calculated from equation (65): $-n=c/2+\sqrt{2}\,mw$. Inverting this formula results in an expression for w:

$$w=-m\left\{\sqrt{2}\,(n+1/2)+\left[(n+1/2)^2+k_3^2+1/4\right]^{1/2}\right\} \tag{67}$$

With the definition of m in equation (62) we conclude the following:

(A) $w>0$ for $k_2<0$ and $w<0$ for $k_2>0$. With equation (57) this implies that positive frequency ($w>0$) waves travel in the negative x^2 direction and negative frequency waves travel in the positive x^2 direction.

(B) One has $|w_n|>\sqrt{2}$, equality for $n=0$, $k_3=0$. Thus there is a minimum frequency for scalar waves in the Gödel universe unless the field does not depend on x^2 (i.e., k_2 is zero). However, in this latter case w and k_3 are also zero, so no wave solution exists.

[15]See equation (A.3b) for the definition of $(c)_n$.

(C) For fixed k_3, w is discrete. This is equivalent to the statement made earlier for the scalar field solutions using the coordinates of equation (17).

(D) There is a straightforward generalization to the massive or conformally invariant cases. The field equation then takes the form

$$g^{\mu\nu}\tilde{\phi}_{;\mu\nu} + A\tilde{\phi} = 0 \tag{68}$$

with $A = m^2$ for a scalar field of mass m, or $A = R/6$ for the conformally invariant scalar field. For the Gödel metric equation (1), the Ricci scalar R has the value $R = +1$. With the additional term in equation (68), equation (56) is modified by $+A\tilde{\phi}$ on the left-hand side. This has the effect of replacing $w^2 + k_3^2$ by $w^2 + k_3^2 + A$ in formulas (58) through (67). The minimum frequency is now larger: $|w| > (1/2)^{1/2} + (1/2 + A)^{1/2}$ instead of (B) above. This reduces to $|w| > \sqrt{2}$ for $A = 0$.

With an explicit formula for the eigenfrequencies w, we plot the spectrum or dispersion relation in Figure 3. The w versus k_3 curves are rectangular hyperbolas. This is best seen by rewriting equation (67) as

$$\left[w + m\sqrt{2}\,(n + 1/2)\right]^2 - k_3^2 = (n + 1/2)^2 + 1/4 \tag{69}$$

The asymptotes of the hyperbolas are centered in the w, k_3 plane at $(k_3, w) = [0, -m\sqrt{2}\,(n + 1/2)]$. The point on each hyperbola closest to the k_3 axis is given by

$$\begin{aligned} w(k_3 = 0) &= -m\left\{\sqrt{2}\,(n + 1/2) + \left[(n + 1/2)^2 + 1/4\right]^{1/2}\right\} \\ &\approx -m\left(\sqrt{2} + 1\right)(n + 1/2) \end{aligned}$$

Equation (62) is in the self-adjoint form of equation (48), and we have imposed regular boundary condition at $z = 0$ and z approaches infinity. Thus we can address the question of completeness of the solutions (64). The explicit form of these solutions is given by

$$f(z) = A\exp(-z/2) z^{\mu+1} L_n^{(2\mu)}(z) \tag{70}$$

with

$$\mu = \left(w^2 + k_3^2 + 1/4\right)^{1/2}$$

These functions can be shown to be orthogonal but incomplete under the inner product $(f, g) = \int f(z) g^*(z) r(z)\, dz$ as follows. Consider the equation

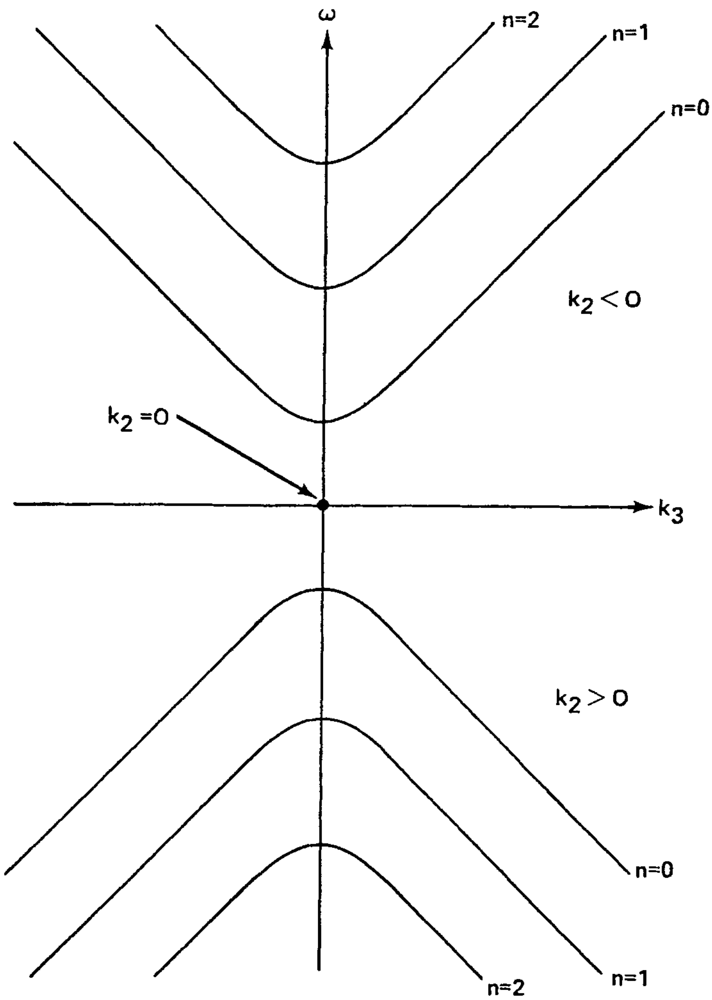

Fig. 3. Spectrum for the scalar field in the Gödel universe.

(62) for f with $-w$ as the eigenvalue and, by equation (48),

$$r(z) = +\sqrt{2}\, m/z = \exp(x^1)/(2k_2) \tag{71}$$

as the weighting function. $w^2 + k_3^2 = \mu^2 - 1/4$ is then the parameter which one holds constant.[16] For each value of $\mu^2 - 1/4$ one can solve equation (62)

[16]One can vary k_3 freely so long as we are considering the z dependence independently of the x^3 dependence.

for the eigenfunctions f and the eigenvalues w. Thus orthogonality of the eigenfunctions holds:

$$\int f^*(x^1, k_2, k_3, w) f(x^1, k_2, k_3, w') r(z)\, dz = |A|^2 \tag{72}$$

However, $|w|$ has an upper bound: $0 < w^2 < \mu^2 - 1/4$. This demonstrates incompleteness for the set of solutions f since a complete set would have $-w$ ranging to infinity.

In the next section we consider the solutions to the neutrino field equations in the Gödel universe.

6. THE NEUTRINO FIELD

The neutrino field equations are solved in two sets of coordinates. The motivation again is to help understand the problems of constructing a field theory, discussed in the next section.

The generalized Dirac equation for curved space-time is (Brill and Wheeler, 1957)

$$-i\gamma^k \nabla_k \psi + m\psi = 0 \tag{73}$$

where $\nabla_k = \partial_k - \Gamma_k$ is the covariant spinor derivative. The spin connections are given by

$$\Gamma_k = -(1/4)([i, kj] + C_{ikj})(\gamma^i\gamma^j - \gamma^j\gamma^i)/2 \tag{74}$$

The $[i, kj]$ are the Christoffel symbols of the first kind[17] and the C_{ikj} are defined by

$$\partial_k \gamma^i = C^i_{jk} \gamma^j \tag{75}$$

To satisfy the anticommutation relations

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij} \tag{76}$$

the γ matrices are chosen to have the form

$$\gamma^0 = \tilde{\gamma}^0 + 2\tanh\theta\,\tilde{\gamma}^1, \qquad \gamma^2 = \tilde{\gamma}^2$$
$$\gamma^1 = \tilde{\gamma}^1/\cosh\theta, \qquad \gamma^3 = \tilde{\gamma}^3 \tag{77}$$

[17]See equation (97), or any standard text on relativity, e.g., Anderson (1967).

The $\tilde{\gamma}^\mu$ are the flat space Dirac matrices:

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \tilde{\gamma}^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{78}$$

From the preceding equations one obtains the following expressions for the spin connections Γ_k:

$$\Gamma_0 = \tilde{\gamma}^1\tilde{\gamma}^2/2\sqrt{2}\,, \qquad \Gamma_i = \cosh\theta\,\tilde{\gamma}^0\tilde{\gamma}^2/2\sqrt{2}$$

$$\Gamma_2 = -\tilde{\gamma}^0\tilde{\gamma}^1/2\sqrt{2}\,, \qquad \Gamma_3 = 0 \tag{79}$$

For the solutions of the Dirac equation (73), simultaneous eigenmodes can be defined by imposing periodicity of the modes along a set of commuting Killing vector fields:

$$L_{\eta_i}\psi = -i\alpha_i\psi, \qquad i = 0,1,3 \tag{80}$$

The Killing vectors are given by equation (8), in the coordinates of equation (9). For the neutrino field one has, in addition to setting $m = 0$ in the above, the restriction to left-handedness:

$$(1 - i\gamma^5)\psi = 0 \quad \text{with} \quad \gamma^5 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{81}$$

$(1 - i\gamma^5)$ is the operator which projects out the right-handed component of the wave function ψ. The neutrino field in this case simplifies to a two-component wave function:

$$\psi = \begin{pmatrix} 0 \\ \tilde{\phi} \end{pmatrix} \quad \text{with} \quad \tilde{\phi} = \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} \tag{82}$$

With equations (77), (79), and (80), the Dirac equation for neutrinos separates in the coordinates of the metric (17). This yields the following equation in θ:

$$\Big[\big(\tilde{\gamma}^0 + \sqrt{2}\tanh\theta\,\tilde{\gamma}^1\big)(-i\alpha_0) + \tilde{\gamma}^1(-i\alpha_1)/\cosh\theta + \tilde{\gamma}^2 d/d\theta$$

$$+\tilde{\gamma}^3(-i\alpha_3) + \tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2/2\sqrt{2} + \tanh\theta\,\tilde{\gamma}^2/2\Big]\psi(\alpha_i,\theta) = 0 \tag{83}$$

To obtain equation (83) the neutrino field ψ has been written in the form, with separated variables,

$$\psi(\alpha_i, x^\mu) = \exp(-i\alpha_0 t - i\alpha_1\phi - i\alpha_3 x)\psi(\alpha_i, \theta) \tag{84}$$

This was possible because of the relation between the three Killing vectors used in equation (80) and the coordinates t, ϕ, and x.

One then obtains coupled equations for the component functions of the neutrino field, $\tilde{\phi}_1$ and $\tilde{\phi}_2$:

$$\left[d/d\theta + \tanh(\theta/2) + \left(\alpha_1 + \sqrt{2}\,\alpha_0 \sinh\theta\right)/\cosh\theta\right]\tilde{\phi}_2 + \left(\alpha_3 + \sqrt{2}/4 - \alpha_0\right)\tilde{\phi}_1 = 0 \tag{85a}$$

$$\left[d/d\theta + \tanh(\theta/2) - \left(\alpha_1 + \sqrt{2}\,\alpha_0 \sinh\theta\right)/\cosh\theta\right]\tilde{\phi}_1 + \left(\alpha_3 + \sqrt{2}/4 - \alpha_0\right)\tilde{\phi}_2 = 0 \tag{85b}$$

These are equivalent to the second-order differential equations:

$$d^2\tilde{\phi}_i/d\theta^2 + \tanh\theta\, d\tilde{\phi}_i/d\theta + \left\{\alpha_0^2 - \left(\alpha_3 + \sqrt{2}/4\right)^2 + (1/\cosh^2\theta) \times\left[1/2 + \sinh^2(\theta/4) - \left(\alpha_1 + \sqrt{2}\,\alpha_0 \sinh^2\theta\right)^2 \pm \left(\alpha_1 \sinh\theta - \sqrt{2}\,\alpha_0\right)\right]\right\}\tilde{\phi}_i = 0,$$

$$+ : i = 1, \qquad - : i = 2 \tag{86}$$

In terms of the variable $y = \sinh\theta$, equation (86) takes the form

$$(d/dy)\left[(1+y^2)\,d\tilde{\phi}_i/dy\right] + \left\{-\left[\left(\alpha_3 + \sqrt{2}/4\right)^2 + \alpha_0^2 - 1/4\right] + \left[2\alpha_0^2 - \alpha_1^2 + 1/4 \pm \sqrt{2}\,\alpha_0 - y\left(2\sqrt{2}\,\alpha_0\alpha_1 \pm \alpha_1\right)\right]\Big/(1+y^2)\right\}\tilde{\phi}_i = 0,$$

$$+ : i = 2, \qquad - : i = 1 \tag{87}$$

This is written in self-adjoint form for comparison with the general self-adjoint differential equation, given in equation (48).

One can compare equation (87) with the corresponding equation for the scalar field case, equation (38). The differences are due to the coupling of the neutrino spin to the metric, as expressed by the presence of the spin connection in the Dirac equation. Equations (87) can be written in the same form as equation (39), but with the k_i given by

$$k_1 = \alpha_0^2 + \left(\alpha_3 + \tfrac{1}{2}\sqrt{2}\right)^2 - \tfrac{1}{4} \qquad k_2 = 2\alpha_0^2 - \alpha_1^2 + \tfrac{1}{4} \pm \sqrt{2}\,\alpha_0$$

$$k_3 = 2\sqrt{2}\,\alpha_0\alpha_1 \pm \alpha_1, \qquad + : i = 2, \qquad - : i = 1 \tag{88}$$

The discussion regarding writing $\tilde{\phi}_i$ in terms of the hypergeometric function and on orthogonality and completeness of the eigenfunctions from equation (39) to equation (53) applies also to the neutrino field, with equation (40) replaced by equation (88). One cannot keep k_1 and k_3 fixed if k_2 is chosen as eigenvalue without changing α_0, α_1, and α_3. If $-k_1$ is chosen as eigenvalue, the discussion of completeness of the neutrino field mode solutions on a three-dimensional surface in the Gödel universe is meaningful, as discussed in Section 8. Again, as with the scalar field, the allowable solutions form a set of measure zero in the complete set of solutions. There are now two differential equations and two sets of eigenfunctions related by the first-order equations (85).

To determine the nature of the eigenvalue spectrum, we consider the form of the asymptotic solutions. However, the boundary conditions must be dealt with differently than for the scalar field case. The neutrino field is a spinor, whereas we require a true scalar quantity, which is independent of any spinor basis. The spinor basis may diverge or vanish at infinity, so that the vanishing or divergence of the spinor field components may be due merely to poor choice of basis. The scalar quantities $\bar{\psi}\gamma_\mu\psi\lambda_i^\mu$ are required to be finite. $\bar{\psi} = \psi^+\tilde{\gamma}^0$ is the Dirac adjoint. λ_i^μ is an orthonormal tetrad[18] $(i = 0, 1, 2, 3)$ parallel transported to infinity along θ-coordinate lines according to

$$d\lambda_i^\mu/ds + \Gamma^\mu_{\alpha\beta}(dx^\alpha/ds)\lambda_i^\beta = 0 \tag{89}$$

with $x^\alpha = (A, B, \theta, C)$; A, B, C constants. $\Gamma^\mu_{\alpha\beta}$ is the connection for the

[18] A tetrad is a set of four vector fields which relate the coordinates to a locally inertial frame at each point. See, e.g., Weinberg (1972).

metric.[19] The tetrad is chosen so that at $\theta = 0$ it has the components

$$\lambda_i^\mu = \delta_i^\mu \qquad \text{at } \theta = 0 \tag{90}$$

The solution to equation (89) with the condition (90) yields the tetrad for all θ as

$$\begin{aligned}
\lambda_0^\mu &= \left(\cosh(\theta/\sqrt{2}) - \sqrt{2}\tanh\theta\sinh(\theta/\sqrt{2}), -\sinh(\theta/\sqrt{2})/\cosh\theta, 0, 0\right) \\
\lambda_1^\mu &= \left(-\sinh(\theta/\sqrt{2}) + \sqrt{2}\tanh\theta\cosh(\theta/\sqrt{2}), -\cosh(\theta/\sqrt{2})/\cosh\theta, 0, 0\right) \\
\lambda_2^\mu &= (0,0,1,0) \\
\lambda_3^\mu &= (0,0,0,1)
\end{aligned} \tag{91}$$

The corresponding scalar quantities of interest are

$$\begin{aligned}
\bar{\psi}\gamma^\mu\psi\lambda_\mu^0 &= \cosh(\theta/\sqrt{2})(\tilde{\phi}_1^*\tilde{\phi}_1 + \tilde{\phi}_2^*\tilde{\phi}_2) + \sinh(\theta/\sqrt{2})(\tilde{\phi}_1^*\tilde{\phi}_2 + \tilde{\phi}_2^*\tilde{\phi}_1) \\
\bar{\psi}\gamma^\mu\psi\lambda_\mu^1 &= -\sinh(\theta/\sqrt{2})(\tilde{\phi}_1^*\tilde{\phi}_1 + \tilde{\phi}_2^*\tilde{\phi}_2) - \cosh(\theta/\sqrt{2})(\tilde{\phi}_1^*\tilde{\phi}_2 + \tilde{\phi}_2^*\tilde{\phi}_1) \\
\bar{\psi}\gamma^\mu\psi\lambda_\mu^2 &= i(\tilde{\phi}_1^*\tilde{\phi}_2 - \tilde{\phi}_2^*\tilde{\phi}_1) \\
\bar{\psi}\gamma^\mu\psi\lambda_\mu^3 &= -\tilde{\phi}_1^*\tilde{\phi}_1 + \tilde{\phi}_2^*\tilde{\phi}_2
\end{aligned} \tag{92}$$

These are required to be finite as θ approaches plus or minus infinity.

The asymptotic form of the equations for $\tilde{\phi}_1$ and $\tilde{\phi}_2$, equations (85a) and (85b), give the values of $\tilde{\phi}_1$ and $\tilde{\phi}_2$ as θ approaches infinity. The requirement of finiteness of the scalars in equation (92) for all values of θ is equivalent to

$$\exp(\pm\theta/\sqrt{2})|\tilde{\phi}_i|^2 \qquad \text{finite as } \theta \to \pm\infty, \text{ respectively} \tag{93}$$

The asymptotic forms of the solutions to equation (87), or equivalently to equation (39), are given, with the appropriate k_i, by equation (57). This gives the requirement

$$B = 0 \quad \text{and} \quad C = 0 \qquad \text{for } \alpha > (1 - 1/\sqrt{2})/2 \tag{94}$$

[19] In a space-time without torsion, such as we consider here, the connections reduce to the Christoffel symbols of the second kind. These are given by, e.g., Anderson (1967).

with $\alpha = +(1/4+k_1)^{1/2}$. This will be the case for only isolated values of $-k_1$ (with k_2, k_3 fixed). However, for $\alpha \leqslant (1-1/\sqrt{2})/2$ no such restriction is implied by equation (93). Thus the spectrum of eigenvalues for k_1 is continuous for $0 \leqslant \alpha \leqslant (1-1/\sqrt{2})/2$ and discrete for $\alpha > (1-1/\sqrt{2})/2$.

Again, as with the scalar field, solution of the neutrino field equations in Gödel's coordinates, equation (1), gives neater results (i.e., explicit mode frequencies) than in the coordinates of equation (17). The Dirac equation is given by equations (73)–(76). In Gödel coordinates we can take

$$\gamma^0 = \tilde{\gamma}^0 - \sqrt{2}\,\tilde{\gamma}^2, \qquad \gamma^1 = \tilde{\gamma}^1, \qquad \gamma^2 = \sqrt{2}\exp(-x^1)\tilde{\gamma}^2, \qquad \gamma^3 = \tilde{\gamma}^3 \quad (95)$$

where $\tilde{\gamma}^i$ are the flat space Dirac matrices given by equation (78). With these γ^i, one calculates the C^i_{jk} as

$$C^2_{21} = -1 \qquad \text{all others zero} \quad (96a)$$

or, equivalently, one finds for the C_{ijk}

$$C_{021} = -\exp(x^1), \qquad C_{221} = -\exp(2x^1)/2 \qquad \text{others zero} \quad (96b)$$

The Christoffel symbols $[i, kj]$ will be given here explicitly for illustration. They are given by derivatives of the metric functions $g_{\mu\nu}$:

$$[i, kj] = \left(dg_{ik}/dx^j + dg_{ij}/dx^k - dg_{kj}/dx^i\right)/2 \quad (97)$$

giving

$$[1,02] = -[2,01] = -[0,12] = -\exp(x^1)/2$$
$$[1,22] = -[2,12] = -\exp(2x^1)/2 \qquad \text{all others zero} \quad (98)$$

Then one can calculate the spin connections Γ_k to be given by

$$\Gamma_0 = \tilde{\gamma}^1\tilde{\gamma}^2/2\sqrt{2}, \qquad \Gamma_1 = 0, \qquad \Gamma_2 = \exp(x^1)\tilde{\gamma}^1\tilde{\gamma}^2/4\sqrt{2}, \qquad \Gamma_3 = 0 \quad (99)$$

For neutrinos one has the condition on the allowed helicity, equation (81). Thus one can write ψ as in equation (82). The neutrino equation is now written out explicitly. Since x^0, x^2, and x^3 do not appear explicitly, we directly separate variables:

$$\tilde{\phi}_i = \exp\left[i\left(k_2x^2 + k_3x^3 - wx^0\right)\right] f_i, \qquad i = 1,2 \quad (100)$$

The neutrino equation then separates to give first-order coupled equations for f_1 and f_2:

$$-i\left(w-k_3-\sqrt{2}/4\right)f_1+\left[d/dx^1+\tfrac{1}{4}+\sqrt{2}\left(w+k_2\exp(-x^1)\right)\right]f_2=0$$

$$-i\left(w+k_3+\sqrt{2}/4\right)f_2+\left[d/dx^1+\tfrac{1}{4}-\sqrt{2}\left(w+k_2\exp(-x^1)\right)\right]f_1=0 \tag{101}$$

These combine to give the second-order differential equations

$$d^2f_i/(dx^1)^2+\tfrac{1}{2}df_i/dx^1+\Big\{w^2-\left(k_3+\sqrt{2}/4\right)^2 + \tfrac{1}{16}-2\left[w+k_2\exp(-x^1)\right]^2\pm\sqrt{2}\,k_2\exp(-x^1)\Big\}f_i=0 \tag{102}$$

with + for f_1 and − for f_2.

For the case $k_2=0$ we can write $f=\exp(px^1)$ since the coefficients in the differential equation (102) are independent of x^1. Then p satisfies the relation

$$p=-1/4\pm\left[w^2+\left(k_3+\sqrt{2}/4\right)^2\right]^{1/2} \tag{103}$$

with f_1, f_2 proportional to $\exp(p_+x^1), \exp(p_-x^1)$. The solutions diverge as x^1 approaches plus infinity or minus infinity unless p is zero. This occurs over a narrow range of w and k_3:

$$w^2+\left(k_3+\sqrt{2}/4\right)^2=1/16 \qquad \text{for } k_2=0 \tag{104}$$

Contrast this with the scalar field case, equation (62), where the $k_2=0$ case gives no wave solution. This is in accord with the discussions based on the solutions in the t, ϕ, θ, x coordinates. There, a continuous spectrum for low-frequency modes [after equation (94)] was found for the neutrino field but not for the scalar field. One finds from equation (101) the ratio of f_1 and f_2:

$$f_1=if_2\left(\sqrt{2}\,w+\tfrac{1}{4}\right)/\left(w-k_3-\sqrt{2}/4\right)=\text{const} \tag{105}$$

For k_2 nonzero define the variable z by

$$z=|k_2|\exp(-x^1) \tag{106}$$

This differs from the scalar field z, equation (63), by a factor of $2\sqrt{2}$. The relations between the x^1 and z derivatives are given by

$$d/dx^1 = -zd/dz, \qquad d^2/(dx^1)^2 = z^2d^2/dz^2 + zd/dz \tag{107}$$

Equation (102) in terms of z becomes

$$z^2d^2f_i/dz^2 + (z/2)df_i/dz + \left[-2z^2 + m\left(\sqrt{2}\,s - 4w\right)z + a\right]f_i = 0 \tag{108}$$

with $a = 1/16 - w^2 - (k_3 + \sqrt{2}/4)^2$; $s = +$ for f_1, $s = -$ for f_2.

Equation (108) is put in the standard form for a second-order linear differential equation with one regular and one irregular singular point[20]:

$$d^2f/dz^2 + p(z)df dz + q(z)f = 0 \tag{109}$$

with

$$p = (1 - \lambda - \lambda')/z, \qquad q = -k^2 + 2\alpha/z + \lambda\lambda'/z^2$$

Comparison of equation (109) with equation (108) yields the values of k, λ, λ', and α:

$$\lambda = \tfrac{1}{4} + \left[w^2 + \left(k_3 + \sqrt{2}/4\right)^2\right]^{1/2}$$

$$\lambda' = \tfrac{1}{4} - \left[w^2 + \left(k_3 + \sqrt{2}/4\right)^2\right]^{1/2}$$

$$k^2 = 2$$

and

$$2\alpha = m\left(\sqrt{2}\,s - 4w\right) \tag{110}$$

The general solution of equation (109) about $z = 0$ is given by

$$\begin{aligned} f = {} & A\exp(-kz)z^{\lambda}F((1 + \lambda - \lambda')/2 - \alpha/k, 1 + \lambda - \lambda', 2kz) \\ & + B\exp(-kz)z^{\lambda'}F((1 - \lambda + \lambda')/2 - \alpha/k, 1 - \lambda + \lambda', 2kz) \end{aligned} \tag{111}$$

$F(a, c, z)$ is the confluent hypergeometric function [see equation (A.3)].

[20]See Appendix; also Morse and Feschbach (1953), pp. 550 ff.

Requiring convergence of the scalars $\bar{\psi}\gamma^{\mu}\psi\lambda^{i}_{\mu}$ is equivalent to convergence of f_1 and f_2 for $0<z<\infty$. The arguments are similar to those presented for the solutions in t, ϕ, θ, x coordinates which resulted in the requirement equation (93) for $\tilde{\phi}_1$ and $\tilde{\phi}_2$.

For z approaching zero, A and B can be nonzero in equation (111) without destroying convergence if $\lambda'>0$ [see equation (A.5a)]. Convergence of f in equation (111) for z approaching infinity, additionally, results in the requirements[21]

$$B=0: \alpha/k=n+\tfrac{1}{2}+\left[w^2+\left(k_3+\sqrt{2}/4\right)^2\right]^{1/2} \tag{112a}$$

for $\lambda'>0$ or $\lambda'\leqslant 0$

$$A=0: \alpha/k=n+1/2-\left[w^2+\left(k_3+\sqrt{2}/4\right)^2\right]^{1/2} \tag{112b}$$

for $\lambda'>0$. The Appendix demonstrates why both alternatives result in convergence for $\lambda'>0$, whereas only equation (112a) results in convergence for $\lambda'\leqslant 0$. α and k are given by equations (112a), (112b), and (110). This yields the possible frequencies of convergent solutions f_1 and f_2:

$$w=[s-2m(n+1/2)]/\sqrt{2}\pm m\left\{[s-2m(n+1/2)]^2/4 + \left(k_3+\sqrt{2}/4\right)^2\right\}^{1/2}$$

with

$$\begin{array}{lll} - \text{ for } \lambda'>0 & \text{or} \quad \lambda'\leqslant 0 & [\text{case (112a)}] \\ + \text{ for } \lambda'>0 & & [\text{case (112b)}] \end{array} \tag{113}$$

We consider the case of the minus sign in equation (113) first, which holds for all λ'. For the case $k_2>0$ ($m=\pm 1$), the frequencies for f_1 and f_2 are given by

$$\begin{aligned} w&=-\left\{\sqrt{2}\,n_1+\left[n_1^2+\left(k_3+\sqrt{2}/4\right)^2\right]^{1/2}\right\} && \text{for } f_1 \\ w&=-\left\{\sqrt{2}\,(n_2+1)+\left[(n_2+1)^2+\left(k_3+\sqrt{2}/4\right)^2\right]^{1/2}\right\} && \text{for } f_2 \end{aligned} \tag{114}$$

[21] $1-\lambda'+\lambda\neq -m$ is identically satisfied and for $\lambda'>0$, $1+\lambda'-\lambda\neq -m$ also always holds. See also equations (A.6).

For $k_2 < 0$ ($m = -1$), one has

$$w = \sqrt{2}\,(n_1 + 1) + \left[(n_1+1)^2 + \left(k_3 + \sqrt{2}/4\right)^2\right]^{1/2} \qquad \text{for } f_1$$

$$w = \sqrt{2}\,n_2 + \left[n_2^2 + \left(k_3 + \sqrt{2}/4\right)^2\right]^{1/2} \qquad \text{for } f_2 \tag{115}$$

Here n_1 and n_2 are the integers required in equations (112) for the convergence of f_1 and f_2, respectively. There is no *a priori* reason why n_1 and n_2 should be the same.

Since any solution pair f_1, f_2 have the same frequencies w, k_2, and k_3, equations (114) and (115) imply the relation

$$n_1 = n_2 + m \tag{116}$$

m is the sign of k_2, as before. The preceding relation is derived in the Appendix together with the relation between f_1 and f_2.

The case $\lambda' > 0$ allows the other sign in equation (113). However, together with equation (116), one can show that the minimum frequency in equation (113) violates the condition $\lambda' > 0$; i.e., $|w|_{\min} = \sqrt{2} - 1$ gives $\lambda'_{\max} = 1/4 - |w|_{\min} < 0$. Thus, only A is nonzero in equation (111) for all w, k_2, and k_3.

The explicit form of the z eigenfunctions can now be given:

$$f_1(z) = A_1 \exp(-2z) z^{\lambda} L_{n_1}^{(c-1)}(4z)$$

$$f_2(z) = A_2 \exp(-2z) z^{\lambda} L_{n_2}^{(c-1)}(4z) \tag{117}$$

The ratio of A_1 and A_2 is given in equations (A.13); λ is given by equation (110), and c by

$$c - 1 = \lambda - \lambda' = 2\left[w^2 + \left(k_3 + \sqrt{2}/4\right)^2\right]^{1/2}$$

The neutrino field ψ is then given by equations (82) and (100), with z given by equation (106) and w by equation (114) or equation (115), depending on m, the sign of k_3.

To further examine the dispersion relations, equations (114) and (115), we cast them in the form

$$\left(w + m\sqrt{2}\,n\right)^2 - \left(k_3 + \sqrt{2}/4\right)^2 = n^2, \qquad n = 1,2,3\ldots \tag{118}$$

This is a rectangular hyperbola with asymptotes given by

$$w + m\sqrt{2}\,n = \pm\left(k_3 + \sqrt{2}/4\right) \tag{119}$$

The "center" (crossing point of the asymptotes) has coordinates $(k_3, w) = (-\sqrt{2}/4, -m\sqrt{2}\,n)$, which shift progressively away from the k_3 axis as n increases. The dispersion relation, or spectrum, is plotted in Figures 4a and 4b. Equations (114) and (115) include only the branch of the hyperbola directed away from the k_3 axis. These are the $\lambda' < 0$ modes. The $\lambda' > 0$

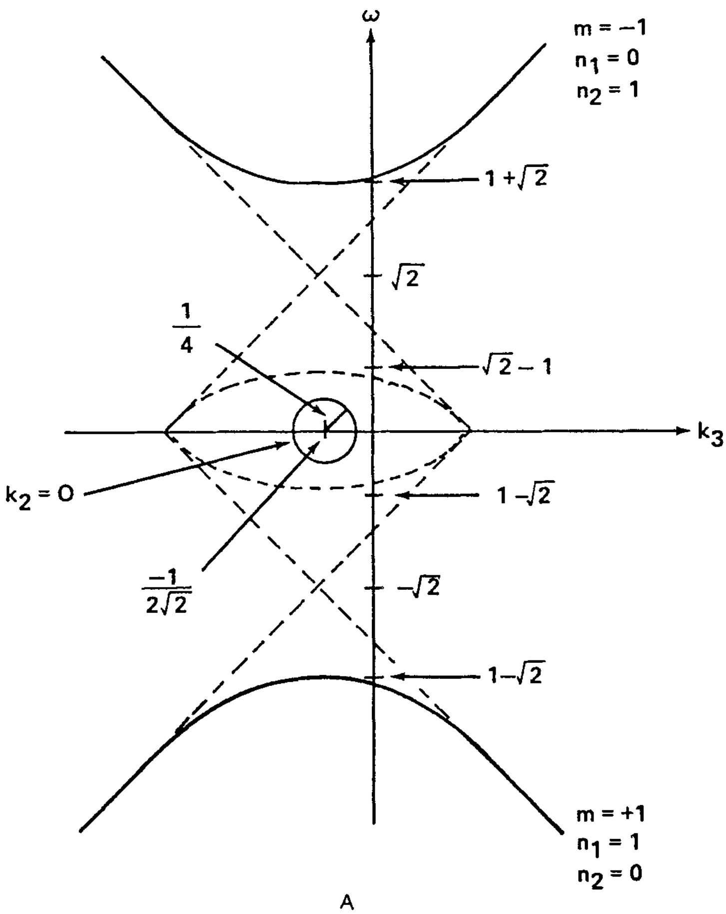

Fig. 4. (a) Spectrum for the neutrino field in the Gödel universe: near the k_3, w origin, illustrating lack of $\lambda' > 0$ modes.

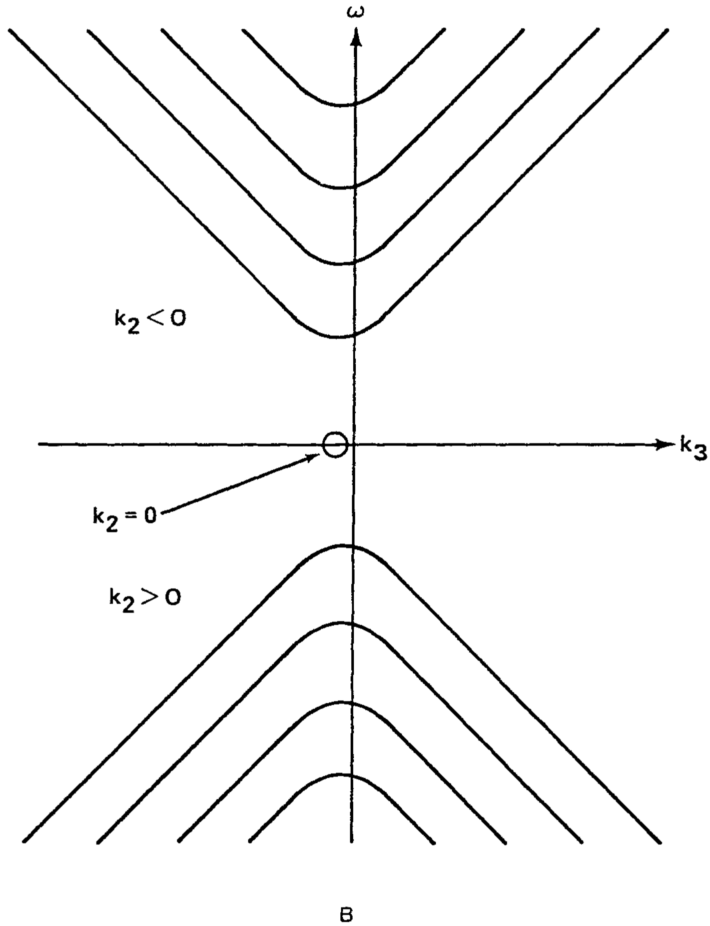

Fig. 4. (b) Spectrum for the neutrino field in the Gödel universe: large-scale features.

modes are excluded as argued in leading up to equation (117). This is indicated in Figure 4a by the branches of the hyperbolas in dotted lines. The hyperbolas for w versus k_3 have asymptotes which recede from the k_3 axis as $-m\sqrt{2}\,n$, but the closest point on each hyperbola recedes as $-m(\sqrt{2}+1)n$. This is seen directly from equation (119) with $k_3+\sqrt{2}/4=0$. Also shown in Figure 4 are the $k_2=0$ modes. This is the circle near the origin of the (k_3, w) plane.

We now summarize the main results obtained via the solution of the neutrino field equations in the x^0, x^1, x^2, x^3 coordinates.

(A) One has $w>0$ for $k_2<0$ and $w<0$ for $k_2>0$, so that positive frequency waves travel in the negative x^2 direction and vice versa. This result was also found for the scalar field.

(B) There is a minimum frequency for neutrino waves (with $k_2 \neq 0$) in the Gödel universe: $|w|>1+\sqrt{2}$.

(C) For fixed k_3, w is discrete, except for neutrino waves with no x^2 dependence ($k_2=0$).

One can show that the solutions f_1 and f_2 do not form a complete set for functions of z in the interval 0 to infinity. To address the question of completeness we write equation (109) in the self-adjoint form [with $p(z)=1/2z$]:

$$(d/dz)(z^{1/2}dx/dz)+z^{1/2}q(z)f=0 \tag{120}$$

with

$$q(z)=-2+2\alpha/z+a/z^2$$

a and α are given by equations (108) and (110). In equation (120) we take α, or equivalently w, as the eigenvalue for fixed values of the parameter a. But by equation (108) for a, one has

$$w^2<1/16-a$$

With this restriction, for any given value of a, w has an upper limit. This is contrary to the requirement for a complete set of eigenfunctions, i.e., that the eigenvalue range to plus infinity. We see thus that the eigenfunctions of equation (117) do not form a complete set. This is in agreement with the results found for the functions $\tilde{\phi}_i$ [discussed after equation (88)].

7. CONSTRUCTION OF A FIELD THEORY IN THE GÖDEL METRIC

Now we wish to consider the question of constructing a field theory for the scalar and neutrino fields in the Gödel metric. There are two parts to the approach followed here. First, the classical field theory is considered. This includes primarily a discussion of the initial-value problem in the Gödel metric. Secondly, the quantization of the fields is briefly discussed. This is not considered in detail owing to the problems encountered with constructing a classical field theory.

So far we have found sets of mode solutions to the scalar and neutrino field equations (Sections 5 and 6, respectively). The equations were separated and solved in both the Gödel (x^0, x^1, x^2, x^3) coordinates and the new coordinates (t, ϕ, θ, x) constructed in Section 2. The equations for the θ and x^1 (or z) dependence were put into self-adjoint form and solved. For both neutrino and scalar fields the θ-dependent functions (or z-dependent functions) were found to be incomplete as a basic set for expanding arbitrary functions of θ (or z). Completeness is defined in terms of an inner product, as was described in the discussion following equations (48a) and (48b).

We need to consider the solutions of the scalar and neutrino equations over the whole of the Gödel space-time. There exists an abstract vector space which includes all functions in the Gödel universe which are solutions of the scalar field equation (and a separate space for all solutions of the neutrino field equation). The two sets of coordinates, (x^0, x^1, x^2, x^3) and (t, ϕ, θ, x), cover the entire Gödel space-time, so the vector spaces for solutions in either set of coordinates are the same. The mode solutions found here [e.g., $\tilde{\phi}$ of equation (37) or of equation (57), for the scalar field] form a basis for a vector space of solutions (in this case, for the scalar field equation). Does the space spanned by the mode solutions encompass all solutions to the field equation? This can be restated in the form: can any solution be expanded in terms of the mode solutions? An inner product is required to decide when two functions are equal, as described earlier.

Normally, the preceding question regarding completeness has a straightforward affirmative answer. This situation occurs when the Cauchy initial-value problem is well defined in a space-time. It is not for the Gödel universe. The Cauchy problem is well defined when a space-time can be foliated into a set of complete spacelike surfaces (conventionally labeled as constant t surfaces). The field equation (here we consider the massless scalar field as an example) $\Box\tilde{\phi}=0$ is separated into an operator ∇^2 on the spacelike surfaces and an operator d^2/dt^2 for the time dependence. Writing $\tilde{\phi}$ proportional to $\exp(-iwt)$, the spatial equation is

$$\nabla^2\tilde{\phi} + w^2\tilde{\phi} = 0 \tag{121}$$

This is Helmholtz's equation, and the solutions to this (for all w^2) form a complete set for expanding all functions on the spacelike surface. ∇^2 here is symbolic for the covariant Laplace operator. The covariant operator is used so that the surfaces can have any curvature as long as they remain spacelike. The statement about completeness also implies that any function on a spacelike surface forms a valid set of initial data for a solution to the field equation.

The inner product under which the solutions to the field equations form a complete set is defined by an integral of the conserved current J^μ

over a spacelike surface:

$$\langle\tilde{\phi}_1,\tilde{\phi}_2\rangle=(i/2)\int d\sum_{\mu}(-g)^{1/2}J^{\mu} \tag{122}$$

This inner product is particularly useful since it is conserved; i.e., the inner product of two solutions does not depend on which surface is considered in equation (122). For the scalar field, the current operator J^{μ} is

$$J^{\mu}(\tilde{\phi}_1,\tilde{\phi}_2)=g^{\mu\nu}\left[\left(\partial_{\mu}\tilde{\phi}_1^*\right)\tilde{\phi}_2-\tilde{\phi}_1^*\left(\partial_{\mu}\tilde{\phi}_2\right)\right] \tag{123a}$$

For the neutrino field (this holds also for the massive Dirac field) the current is given by

$$J^{\mu}(\psi_1,\psi_2)=\bar{\psi}_1\gamma^{\mu}\psi_2 \tag{123b}$$

Applied to the scalar field case, the inner product, equation (122), becomes

$$\langle\tilde{\phi}_1,\tilde{\phi}_2\rangle=(i/2)\int_{t=\text{const}}d^3x(-g)^{1/2}g^{0\mu}\left[\left(\partial_{\mu}\tilde{\phi}_1^*\right)\tilde{\phi}_2-\tilde{\phi}_1^*\left(\partial_{\mu}\tilde{\phi}_2\right)\right] \tag{124}$$

What alternate inner products exist? One can use

$$(\tilde{\phi}_1,\tilde{\phi}_2)=\int\tilde{\phi}_1^*\tilde{\phi}_2(-g)^{1/2}d^3x \tag{125}$$

This is a generalization from the inner product used for one variable, equation (49a). The inner product in equation (125) is not conserved. Another possibility, perhaps useful when the Cauchy problem is not well defined, is

$$\int\tilde{\phi}_1^*\tilde{\phi}_2(-g)^{1/2}d^4x$$

The mode solutions are then not a complete basis, since any function (including all which do not satisfy the field equations), over the full space-time can be considered. However, if the space is Euclideanized (i.e., t goes to it), then the solutions to the massive scalar field equation: $\Box\tilde{\phi}+m^2\tilde{\phi}=0$ for all values of m form a complete set. Compare this with equation (121); the Euclideanized $\Box$ is now a Laplace operator. The question of what forms a valid and useful inner product for the space of solutions has not received enough attention in the past. It deserves much more attention, but only a brief discussion is given here.

What can go wrong with the standard classical field theory? A simple example involving an incorrect choice of surface for defining the inner product (122) is now given. Consider the massive scalar field in flat space with coordinates (t, x, y, z). Define the inner product as in equation (124) but over $z = \text{const}$ surfaces. The mode solutions are given by

$$\exp(-iwt + ik_x x + ik_y y + ik_z z) \tag{126a}$$

with

$$w^2 - k_x^2 - k_y^2 - k_z^2 - m^2 = 0 \tag{126b}$$

According to equation (126b), the k's can have any value, whereas w is restricted to $|w| > m$. Thus, on a constant t surface, the modes (126a) form a complete set of functions of x, y, and z. But on a constant z surface, the modes (126a) are incomplete for functions of t, x, and y owing to lack of the low-frequency $\exp(-iwt)$ functions. This problem arises because a constant z surface is not spacelike; this is responsible for the plus sign of w^2 occurring in equation (126b).

In the Gödel metric there are no complete spacelike surfaces, so the problems of the last paragraph are expected to occur. In Sections 5 and 6, the incompleteness of the mode solutions for the θ (or the x^1) dependence was demonstrated for both scalar and neutrino fields. This is easily extended to incompleteness over constant t (or x^0) surfaces under either inner product equation (124) or (125) for the scalar field or equation (122) for the neutrino field. It was shown that the mode solutions form a set of measure zero in the total set of basic functions on a constant t (or x^1) surface. This is in contrast to the incompleteness of the modes (126a) over a constant z surface in flat space for which only a few basic functions (those for $|w| < m$) were missing.

The modes (126a) still form a complete basis for all solutions to the scalar field equation for mass m in flat space. For the Gödel metric, we do not know whether the mode solutions form a complete basis for the solutions of the field equation (either scalar or neutrino). This question is one which should be looked into, but is not considered here. Another question, not addressed here either, is whether the mode solutions in the coordinates (t, ϕ, θ, x) span the same vector space as those in the coordinates (x^0, x^1, x^2, x^3).

The next section discusses the lack of completeness of the mode solutions over three-dimensional surfaces in the Gödel metric. If the Cauchy problem were well defined for the Gödel universe, completeness would be essential. Since it is not, completeness in the standard sense is not expected to hold.

8. ORTHOGONALITY AND COMPLETENESS OF THE MODE SOLUTIONS

Here we consider orthogonality and completeness of the full four-dimensional, dependent mode solutions for the scalar field, equation (37). The discussion is dependent on the choice of an inner product for the solutions to the scalar field equation. The lack of completeness is not surprising in light of what has been said in Section 7.

The scalar field modes in (t, ϕ, θ, x) coordinates are examined first. They are given by equation (37), with $\tilde{\phi}(\theta)$ given by equations (45), (46), or (47) for the appropriate values of $y = \sinh\theta$. We first consider k_2 as eigenvalue [alternative (48b)] for the θ (or y) equation. In this case k_1 and k_3 are parameters. The relation between the k's and the α's, equation (40), shows that k_2 cannot be varied freely (as it must be to "find" its eigenvalues for fixed k_1 and k_3) without changing α_0 or α_1. In turn, constant k_3 implies that both α_0 and α_1 change. Constant k_1 means that α_3 also changes. This is unsatisfactory since to discuss completeness we need to have α_1 and α_3 fixed so that the completeness in the variables θ and x is undisturbed [see equation (37)]. The t dependence is not of concern since we can define the inner product in the standard manner as over a $t = \text{const}$ surface, as in equation (124) or equation (125). We seek completeness over a constant t surface in order to decompose any initial data into field modes. The Gödel metric presents an additional problem here since a constant t surface is not spacelike everywhere nor do there exist any complete spacelike surfaces.

The other choice of $-k_1$ as eigenvalue is now considered. With $-k_1$ as eigenvalue and k_2, k_3 fixed parameters for equation (39), the relations (40) leave only one choice. α_0 and α_1 must be fixed and α_3 allowed to vary to change $-k_1$. For consistency with completeness defined for an inner product on a set of three-dimensional surfaces in the Gödel metric (recall there are no complete spacelike surfaces available), one must take $x = \text{const}$ as these surfaces. Then variation of $-k_1$ (and α_3) does not disturb the completeness in t and ϕ variables. This unusual situation in which α_3 takes on the status of energy (normally one would give α_0 this role) and x the status of a pseudo-time-coordinate is forced on us and is a result of the rotation of the Gödel universe.

For the full four-dimensional space-time we then generalize the relation (52b) to obtain, for orthogonality,[22]

$$\int d\theta\, dt\, d\phi \exp[-i\phi(\alpha_1-\alpha_1')-ix(\alpha_3-\alpha_3')-it(\alpha_0-\alpha_0')]$$
$$\times\tilde{\phi}(\alpha_0,\alpha_1,-k_1,\theta)\tilde{\phi}^*(\alpha_0',\alpha_1',-k_1',\theta)\cosh\theta$$
$$=\exp[-ix(\alpha_3-\alpha_3')]\,\delta_{\alpha_1,\alpha_1'}\delta_{|\alpha_3|,|\alpha_3'|}\delta_{\alpha_0,\alpha_0'}\delta_{k_1,k_1'} \qquad (127)$$

If $-k_1$ would cover the full range of eigenvalues, the completeness relation

$$\sum_{\alpha_0,\alpha_1,k_1}\tilde{\phi}(\alpha_0,\alpha_1,-k_1,\theta)\tilde{\phi}^*(\alpha_0,\alpha_1,-k_1,\theta')$$
$$\times\exp[-i\alpha_0(t-t')-i\alpha_1(\phi-\phi')]=\delta(\theta-\theta')\delta(t-t')\delta(\phi-\phi') \qquad (128)$$

would be valid. Since completeness is defined on constant x surfaces, $x=x'$ is required and implied in equation (127). As mentioned previously, $-k_1$ will not take on all eigenvalues for any values of α_0 or α_1. Thus, the completeness relation, equation (128), will not hold.

One can determine the missing eigenfunctions which prevent the solutions to the scalar field equation from forming a complete set. We use $-k_1$ [alternative (48a)] as eigenvalue to discuss the missing eigenfunctions. In this case we had defined an inner product over constant x surfaces instead of constant t surfaces and treated α_3 as the "frequency" of our solutions to be quantized. Then, by equation (40), k_1 varies as α_3^2 with k_2 and k_3 fixed [as well as the t and ϕ dependence through α_0 and α_1 in equation (37)]. There are missing eigenfunctions since $-k_1$ has a possible range of minus infinity to $-\alpha_0^2$, whereas the eigenvalues required range from a minimum value to plus infinity. We see that the high-frequency [$\exp(-i\alpha_3 x), \alpha_3$ large] solutions are the ones that are missing. This is a strange result since x is the ignorable coordinate in the Gödel metric. We note also here that for fixed k_2 and k_3, $-k_1$ takes on only a finite number of the infinite range of eigenvalues. Thus, the set of allowed solutions of the scalar wave equation is of measure zero in the complete set of eigenfunctions.

[22] The $\delta_{|\alpha_3|,|\alpha_3'|}$ is present, but redundant since $(\alpha_0,\alpha_1,-k_1)=(\alpha_0',\alpha_1',-k_1)$ implies $\alpha_3=\pm\alpha_3'$.

The arguments for the eigenmode solutions in (t, ϕ, θ, x) coordinates for the neutrino field equation parallel this. The counterpart of equation (127) has $\delta_{|\alpha_3+(1/2)\sqrt{2}|,|\alpha_3'+(1/2)\sqrt{2}|}$ instead of $\delta_{|\alpha_3|,|\alpha_3'|}$ on the right-hand side. This is due to the difference between equation (40) and equation (88) for k_1 (orthogonality of the ϕ- and t-dependent parts of ψ gives $k_1 = k_1'$ and $\alpha_0 = \alpha_0'$).

One can define an inner product in a way similar to the standard manner by an integral of the conserved current J^μ over a constant t surface. This surface is not spacelike everywhere, in particular for $|\sinh\theta| > 1$. One writes the inner product as in equation (124). The inner product (124) is different from that in equation (127) or its counterpart for k_2 as eigenvalue which arise from equation (49a). Under this inner product one can show that the eigenmodes of equation (37) with different α_0, α_1, or α_3 are orthogonal:

$$\langle \tilde{\phi}(\alpha_0, \alpha_1, \alpha_3; x^\mu), \tilde{\phi}(\alpha_0', \alpha_1', \alpha_3'; x^\mu) \rangle = \delta_{\alpha_0, \alpha_0'} \delta_{\alpha_1, \alpha_1'} \delta_{\alpha_3, \alpha_3'} \tag{129}$$

$\tilde{\phi}$ needs a different normalization than for the other inner product, equation (49a). Here the values of α_3^2 are dictated by the eigenvalues for $-k_1$, giving two possible signs: $\alpha_3 = \pm(k_1 - \alpha_0^2)^{1/2}$.

We demonstrate here, explicitly, the orthogonality of the mode solutions to the scalar field equation in the coordinates (x^0, x^1, x^2, x^3). In Section 5 we found all solutions to the scalar field equation which are regular for x^2 approaching plus or minus infinity (z approaching plus infinity or zero). From equations (57) and (70) the scalar field modes have the form

$$\tilde{\phi}(x^\mu, w, k_2, k_3) = A \exp\left[i\left(k_2 x^2 + k_3 x^3 - w x^0\right)\right] \exp(-z/2) z^{\mu+1} L_n^{(2\mu)}(z)$$

with

$$z = 2\sqrt{2}\,|k_2| \exp(-x^1), \qquad \mu = \left(w^2 + k_3^2 + 1/4\right)^{1/2} \tag{130}$$

and w given by equation (67).

The orthogonality of the $f(x^1)$ eigenfunctions is given by equation (72). Orthogonality of the normal modes of equation (130) is then easily demonstrated to hold:

$$\int \tilde{\phi}^*(x^\mu, k_2, k_3, w) \tilde{\phi}(x^\mu, k_2', k_3', w') r(z)\, d^3x$$
$$= (2\pi)^2 |A|^2 \delta_{w,w'} \delta(k_2 - k_2') \delta(k_3 - k_3') \tag{131}$$

In equation (131) the weighting function r is given by

$$r(z) = -\sqrt{2}\,m/z = -m\exp(x^1)/2|k_2| = -(-g)^{1/2}\big(\sqrt{2}\,k_2\big) \quad (132)$$

This is directly proportional to $(-g)^{1/2}$, so that orthogonality of the scalar field modes is assured under the inner product of equation (125).

We now define an alternate inner product by equation (124), with the integration taken over a constant x^0 surface. The J^0 component of the current operator applied to two normal modes, equation (123a), gives

$$J^0(\tilde{\phi}_1, \tilde{\phi}_2) = \big[-i(w+w') - 2\exp(-x^1)i(k_2+k_2')\big]\tilde{\phi}_1^*\tilde{\phi}_2 \quad (133)$$

Thus, the integral in equation (124) becomes

$$\begin{aligned}\langle\tilde{\phi}_1, \tilde{\phi}_2\rangle &= \int \exp(x^1)\,dx^1\,dx^2\,dx^3/\sqrt{2}\\ &\quad\times\big[(w+w')/2 + 2\exp(-x^1)(k_2+k_2')/2\big]\tilde{\phi}_1^*\tilde{\phi}_2\\ &= (2\pi)^2\delta(k_2-k_2')\delta(k_3-k_3')\int \exp(x^1)\,dx^1\\ &\quad\times\big[(w+w')/2 + 2\exp(-x^1)k_2\big] f^*(x^1, w, k_2, k_3)\\ &\quad\times f(x^1, w', k_2, k_3) \end{aligned} \quad (134)$$

However, from the differential equation for f, equation (58), one has

$$\begin{aligned}&\exp(-x^1)(d/dx^1)\big[\exp(x^1)df/dx^1\big]\\ &\quad= \big[w^2 + k_3^2 + 4\exp(-x^1)wk_2 + 2k_2^2\exp(-2x^1)\big] f \end{aligned} \quad (135)$$

From equation (135) one obtains the following relation:

$$\begin{aligned}&f_1^*(x^1, w', k_2, k_3)(d/dx^1)\big[\exp(x^1)df_2(x^1, w, k_2, k_3)/dx^1\big]\\ &\quad - f_2(d/dx^1)\big[\exp(x^1)df_1^*/dx^1\big]\\ &\quad= \big[w'^2 - w^2 + 4\exp(-x^1)k_2(w'-w)\big] f_1^* f_2 \exp(x^1) \end{aligned} \quad (136)$$

The left-hand side of equation (136) vanishes upon integrating by parts and applying boundary conditions. Thus, the right-hand side must vanish when integrated. But this is just the integrand of equation (134) times $w'-w$. This means that for $w' \neq w$, the right-hand side of (134) must vanish.

Thus with proper normalization for $\tilde{\phi}$, the inner product, equation (124), has the form

$$\langle \tilde{\phi}_1, \tilde{\phi}_2 \rangle = \delta(k_2 - k_2')\delta(k_3 - k_3')\delta_{w,w'}$$

for $\tilde{\phi}_1$ and $\tilde{\phi}_2$ normal modes. The discrete Kronecker delta in w is used since the w's are discrete, corresponding to different values of n in equation (67). The orthogonality of the neutrino modes can be shown similarly with inner product given by equations (122) and (123b).

9. QUANTIZATION OF THE SCALAR AND NEUTRINO FIELDS

Here a brief discussion will be given on the quantization of a field in general. Then the problems associated with attempting to quantize in the Gödel universe are mentioned. As pointed out in Section 7, even classical field theory in the Gödel universe has its problems, so what is said here regarding the quantized field is rather incomplete.

The quantized field is represented by the field operator. The field operator can be expanded as a sum of positive frequency modes times annihilation operators plus negative frequency modes times creation operators.[23] The annihilation and creation operators satisfy commutation relations for the scalar field [see equation (142)] or anticommutation relations for the neutrino field.

The momentum canonically conjugate to the field operator Φ is defined via the Langrangian L:

$$\pi = \delta L / \delta(d\Phi/dt) \tag{137}$$

For the scalar field the conjugate momentum is

$$\pi = (-g)^{1/2} g^{0\mu} \Phi^*_{,\mu} \tag{138}$$

and for the neutrino field one has

$$\pi = i(-g)^{1/2} \Phi \tilde{\gamma}^0 \gamma^0 \tag{139}$$

[23]More details are given in Bjorken and Drell (1965).

The canonical commutation or anticommutation relations take the form, on a constant time surface, in analogy with the classical Poisson brackets:

$$[\Phi(x^\mu), \pi(x^{\mu\prime})]_\pm = i\delta^3(x^\mu - x^{\mu\prime}), \qquad t = t' \tag{140}$$

In equation (140) the $-$ or $+$ subscript indicates use of the commutator for the scalar field or anticommutator for the neutrino field, respectively.

The preceding procedure can be shown to be consistent, i.e., starting with the field operator expansion [e.g., equation (141)] and the commutation relations for the operators [equation (142)], the left-hand side of equation (140) is calculated to be $i(-g)^{1/2}$ times the sum over positive frequency modes (label these by n) of $w_n[\tilde{\phi}_n(x^\mu)\tilde{\phi}_n^*(x^{\mu\prime}) + \tilde{\phi}_n^*(x^\mu)\tilde{\phi}_n(x^{\mu\prime})]$. This reduces, by the completeness relation for eigenfunctions, to the right-hand side of equation (140). The reverse procedure of deriving the commutation relations of the annihilation and creation operators from the field operator expansion and equation (140) is also valid.

We now consider quantization of the scalar field in the Gödel metric. Quantum field theory, in its present form, for any space-time has been based on time evolution of the field from one complete spacelike surface to another, i.e., the Cauchy initial-value problem. We wish to write the field operator in terms of positive and negative frequency modes times annihilation and creation operators, respectively. Since the Gödel universe does not have a foliation into complete spacelike surfaces on which to base a time evolution, we are forced to other means in defining positive and negative frequency. The following discussion is based on the solutions in (x^0, x^1, x^2, x^3) coordinates, since that is more straightforward.

For the scalar field operator Φ, one can make the expansion

$$\Phi(x^\mu) = \int_{k_2<0} dk_2\, dk_3 \sum_n \left[a(k_2, k_3, w_n)\tilde{\phi}(k_2, k_3, w_n; x^\mu)\right.$$
$$\left. + b^+(k_2, k_3, w_n)\tilde{\phi}^*(k_2, k_3, w_n; x^\mu)\right] \tag{141}$$

in terms of the mode solutions of Section 5 to the scalar field equation. These are explicitly given by equation (130). Equation (141) is the expansion for the charged scalar field, rather than for the neutral scalar field which would have a rather than b in the second line above. "b"-type and "a"-type particles carry a scalar charge of opposite sign (see, e.g., Bjorken and Drell, 1965), but this is not essential to the discussion here. The creation and annihilation operators satisfy the commutation relations

$$\begin{aligned}[a(k_2,k_3,w_n),a^+(k_2',k_3',w_n')] &= \delta(k_2-k_2')\delta(k_3-k_3')\delta_{w_n,w_n'}\\ [b(k_2,k_3,w_n),b^+(k_2',k_3',w_n')] &= \delta(k_2-k_2')\delta(k_3-k_3')\delta_{w_n,w_n'}\end{aligned} \qquad (142)$$

All other commutators vanish.

The following inner products (with $w_n > 0$) yield the annihilation and creation operators:

$$\begin{aligned}\langle\tilde{\phi}(k_2,k_3,w_n;x^\mu),\Phi(x^\mu)\rangle &= a(k_2,k_3,w_n)\\ \langle\tilde{\phi}^*(k_2,k_3,w_n;x^\mu),\Phi(x^\mu)\rangle &= b^+(k_2,k_3,w_n)\end{aligned} \qquad (143)$$

The field operator obeys the commutation relation

$$\begin{aligned}[\Phi(x^{\mu\prime}),\pi(x^\mu)] = -i\int_{k_2<0} dk_2\,dk_3 \sum_n &\left[w_n \exp(x^1)/\sqrt{2}+\sqrt{2}\,k_2\right]\\ &x[\tilde{\phi}^*(k_2,k_3,w_n;x^\mu)\tilde{\phi}(k_2,k_3,w_n;x^{\mu\prime})\\ &+\tilde{\phi}(k_2,k_3,w_n;x^\mu)\tilde{\phi}^*(k_2,k_3,w_n;x^{\mu\prime})]\end{aligned} \qquad (144)$$

where π is defined by equation (138). However, the failure of completeness for the mode solutions to the scalar field equation in the Gödel metric prevents the field commutator (144) from reducing to a delta function as in equation (140). Note that the factor on the first line of equation (144), i.e., $w_n \exp(x^1)+2k_2$, is identical to that appearing in equation (134) (setting $w=w'$ and $k_2=k_2'$) for the inner product. Thus, equation (144) contains the correct weighting function, and only the lack of a complete set of modes, as previously demonstrated, prevents equation (144) from reducing to equation (140).

The field operator commutator, equation (140), expresses the independence of any two points on a spacelike surface (one can choose constant time surfaces in any consistent fashion by change of coordinates); i.e., the fields at x^μ and $x^{\mu\prime}$ commute for $x^\mu \neq x^{\mu\prime}$. This is equivalent to freedom in specifying initial data arbitrarily on a Cauchy surface. We do not have this freedom for the Gödel universe. How is this related to the presence of closed timelike lines?

One can construct the following function for the scalar wave equation:

$$H(x^{\mu}; x^{\mu\prime}) = \int_{k_2>0} dk_2\, dk_3 \sum_n \tilde{\phi}(k_2, k_3, w_n, x^{\mu}) \times \tilde{\phi}^*(k_2, k_3, w_n, x^{\mu\prime}) \exp(x^1)/2k_2 \tag{145}$$

This function is based on the modes of Section 5 in terms of the (x^0, x^1, x^2, x^3) coordinates. From an arbitrary function $f(x^{\mu})$, one can construct a solution to the scalar field equation:

$$\tilde{\phi}(x^{\mu\prime}) = \int_{x^0=\text{const}} dx^1\, dx^2\, dx^3 f(x^{\mu}) H(x^{\mu}; x^{\mu\prime}) \tag{146}$$

For $x^0 = x^{0\prime}$, H is not equal to $\delta^3(x^{\mu} - x^{\mu\prime})$ due to the failure of the completeness relation, equation (49b), to hold. In this case we are using the inner product of equation (125). $r(z)$ is given by $\exp(x^1)/2k_2$ from equation (71). Thus, $\tilde{\phi}(x^{\mu})$ is not, in general, equal to $f(x^{\mu})$ even on the initial data $x^0 = \text{const}$ surface. The closed timelike lines may be thought of as linking any $x^0 = \text{const}$ surface with itself. Any initial data must be self-consistent with the field propagated along the closed timelike lines back to the initial data surface. The absence of a global Cauchy surface for the Gödel universe is directly related to the phenomenon of closed timelike lines.

The discussion regarding neutrino fields in this respect is analogous. The analogous H function can be used to demonstrate the lack of freedom of choice of initial data on a constant x^0 surface. The neutrino mode solutions can be shown to be orthogonal under an inner product based on the spinor current $\bar{\psi}\gamma^{\mu}\psi$. In addition, the "equal time" (i.e., $x^0 = x^{0\prime}$) anticommutator of the neutrino field operators at x^{μ} and $x^{\mu\prime}$ fails to give a delta function in x^{μ} and $x^{\mu\prime}$, in analogy to the scalar field case, equation (144). Again, this is due to the incompleteness of the mode solutions over a three-dimensional surface.

10. SUMMARY AND DISCUSSION

An investigation of the neutrino and scalar fields in the Gödel universe has been carried out. The question of the symmetries of the Gödel universe was addressed first and was utilized to construct new coordinates (t, ϕ, θ, x). The geodesics were found and the behavior of light cones examined to illustrate the nature of the closed timelike lines in the Gödel universe.

The massless scalar and neutrino field equations were solved next. The mode functions and eigenfrequencies were found. Positive frequency waves travel in the negative x^2 direction and negative frequency waves in the positive x^2 direction for both scalar and neutrino fields.

The massless scalar and neutrino fields are found to possess quantized (discrete) energy and momenta. For the modes in (t, ϕ, θ, x) coordinates, these are $(\alpha_0, \alpha_1, -k_1, \alpha_3)$. One can regard α_3 as a function of α_0, α_1, and $-k_1$, this function being the dispersion relation (cf. $w^2 = k_x^2 + k_y^2 + k_z^2$ for plane waves in flat space-time). The discreteness or quantization of $-k_1$ is associated with a potential in the θ direction. The functional relation giving the quantization depends on the solution of the hypergeometric equation (39) subject to the proper boundary conditions, with k_i given by equation (40) for the scalar field and by equation (88) for the neutrino field.

The modes in (x^0, x^1, x^2, x^3) coordinates were expressed in terms of generalized Laguerre polynomials by equation (130) for the scalar field and equations (82), (100), and (117) for the neutrino field. The eigenfrequencies were found explicitly as given by equation (67) for the scalar modes and equations (114) and (115) for the neutrino modes, in both cases for $k_2 \neq 0$. For $k_2 = 0$, equations (60) and (104) give the frequencies.

One might expect this discreteness of frequency purely from a consideration of the geodesics for a classical particle: the motion in y [$= \sinh\theta - \sqrt{2}AC/D$, defined by equation (24b)] is that of a particle in a simple harmonic oscillator potential [see equation (25)]. The discreteness would thus occur for the momentum corresponding to the coordinate of the periodic motion.

An unusual result is found for the neutrino field in the low-energy, long-wavelength region [i.e., for $\alpha \leqslant (1 - 1/\sqrt{2})/2$, $\alpha = (1/4 + k_1)^{1/2}$]. The spin gravitation coupling effect becomes large enough to overcome the effective potential. The continuous allowable range of energy and momenta characteristic of an unbound particle is regained.

The neutrino is not invariant under the parity transformation (inversion of spatial coordinates), whereas the scalar particle is. One would expect any effects due to lack of inversion symmetry in the Gödel universe to show up in the neutrino wave functions. Inversion of ϕ and x coordinates is equivalent to the change in the wave function:

$$\alpha_1 \text{ to } -\alpha_1, \qquad \alpha_3 \text{ to } -\alpha_3$$

The change of θ to $-\theta$ is then determined by the differential equation (39) using the preceding relation. For the scalar field, one finds from the relations (40) that k_1, k_2, and $k_3 y$ are unchanged, so that the wave function is invariant under parity. However, the relations (88) show that k_1 and $k_3 y$

change for the neutrino case so that the neutrino field is not invariant. Neither is the neutrino current invariant: $J^\mu = \bar{\psi}\gamma^\mu\psi$ ($\bar{\psi}$ being the Dirac adjoint: $\bar{\psi} = \psi^+ \tilde{\gamma}^0$) since the change in $\tilde{\phi}_i$ is not merely one of phase.

The solutions to the field equations (either scalar or neutrino) do not form a complete set over a three-dimensional surface in the Gödel metric. This is connected with the problem of constructing a field theory in a space-time for which the Cauchy initial-value techniques are inapplicable. More work is required, but it is not clear whether it makes sense to construct a field theory in such a space-time, without drastically altering the standard procedure.

Despite the foregoing problems, a preliminary discussion of quantizing such a field was given. The purpose of this was to point out the major shortcomings of the approach used. The interpretation of quantum mechanics in a universe with closed timelike lines has its problems, over and above the problems associated with causality violation in a classical space-time (which include the failure of Cauchy techniques). They are clearly an area of great interest, but it is believed that such a discussion is beyond the scope of

The second quantization procedure here is, of necessity, incomplete. A field operator expression [equation (141)] and commutation relations [equations (142)] for the annihilation and creation operators were defined in the usual manner. This led to the commutator (144) for the field operator at $x^{\mu\prime}$ and its conjugate momentum at x^μ, which differs from the usual delta function result, equation (140), because of the incompleteness of the mode solutions to the scalar and neutrino field equations. The incompleteness, the lack of a complete Cauchy surface, and the presence of closed timelike lines all stem from the presence of rotation globally in the Gödel universe.

One suspects that standard quantum theory may not make much sense in a space-time which violates causality such as the Gödel universe. The causality violation which occurs in a classical space-time might be always removed by quantum effects. For example, it has been shown that the inner horizon of the charged black hole is unstable (Matzner, Zamorano, and Sandberg, 1979 and references therein). Perturbations in a test field outside the black hole result in infinite energy densities at the inner horizon, strongly indicating that it would be disrupted. The inner horizon is responsible for the causality-violating effects for the charged (and also for the rotating) black hole.

ACKNOWLEDGMENTS

I wish to thank W. G. Unruh for many helpful discussions and the N.S.E.R.C. for partial support.

APPENDIX: SOME PROPERTIES OF THE CONFLUENT HYPERGEOMETRIC FUNCTION

In this Appendix, features of the confluent hypergeometric function are used in calculating some results needed for the solution of the scalar and neutrino fields in the Gödel metric.

Here we wish to consider the solution of the standard differential equation with one regular singular point at $z=0$ and one irregular singular point at infinity. We write the equation in the form

$$d^2f/dz^2+p(z)df/dz+q(z)f=0 \tag{A.1}$$

with

$$p=(1-\lambda-\lambda')/z \quad \text{and} \quad q=-k^2+2\alpha/z+\lambda\lambda'/z^2$$

The general solution of equation (A.1) about $z=0$ is given by

$$\begin{aligned} f(z)=&A\exp(-kz)z^{\lambda}F((1+\lambda-\lambda')/2-\alpha/k,1+\lambda-\lambda',2kz)\\ &+B\exp(-kz)z^{\lambda'}F((1-\lambda+\lambda')/2-\alpha/k,1-\lambda+\lambda',2kz) \end{aligned} \tag{A.2}$$

$F(a,c,z)$ is the solution of the confluent hypergeometric equation which is regular about $z=0$. The standard form of the confluent hypergeometric equation is

$$zd^2F/dz^2+(c-z)dF/dz-aF=0 \tag{A.3a}$$

The solution F is given by

$$F(a,c,z)=\sum_{n=0}^{\infty}(a)_n z^n/((c)_n n!) \tag{A.3b}$$

with $(x)_n$ defined by $(x)_n=x(x+1)\cdots(x+n-1),(x)_0=1$. The confluent hypergeometric function has the limits as z approaches zero or infinity

$$\begin{aligned} &F(a,c,0)=1, \qquad c\neq -n \qquad (n=0,1,2\ldots)\\ &F(a,c,z)\to \exp(z)z^{a-c}\Gamma(c)/\Gamma(a) \qquad \text{for } z\to\infty \end{aligned} \tag{A.4}$$

$\Gamma(x)$ is the gamma function (see, e.g., Abramnowitz and Stegun, 1964).

The solution of equation (A.1) then is found to have the limits

$$f(z) \to Az^{\lambda} + Bz^{\lambda'}, \qquad z \to 0 \tag{A.5a}$$

$$\begin{aligned} f(z) \to {} & A\exp(kz)z^{-(1-\lambda-\lambda')/2-\alpha/k}\Gamma(1+\lambda-\lambda')/\Gamma((1+\lambda-\lambda')/2-\alpha/k) \\ & + B\exp(kz)z^{-(1-\lambda-\lambda')/2-\alpha/k}\Gamma(1-\lambda+\lambda')/ \\ & \times\Gamma((1-\lambda+\lambda')/2-\alpha/k), \qquad z\to\infty \end{aligned} \tag{A.5b}$$

The gamma function $\Gamma(x)$ has simple poles at the negative integers $x = -n$ $(n = 0, 1, 2\ldots)$.

As z approaches zero [equation (A.5a)], $f(z)$ will converge for any values of A and B if λ and λ' are greater than or equal to 0. However, if either λ or λ' is negative, the corresponding constant, A or B, must vanish. For convergence as z approaches infinity, first consider the case when λ and λ' are both greater than or equal to 0. Then one can have convergence if the gamma function in the denominator of equation (A.5b) has a pole (and the gamma function in the numerator does not). The two possibilities are

$$A=0 \quad \text{and} \quad (1-\lambda+\lambda')/2-\alpha/k=-n, \qquad 1-\lambda+\lambda' \neq -m \tag{A.6a}$$

$$B=0 \quad \text{and} \quad (1+\lambda-\lambda')/2-\alpha/k=-n, \qquad 1+\lambda-\lambda' \neq -m \tag{A.6b}$$

If only one of λ or λ' is negative, say λ', then one requires $B=0$ for convergence as z approaches 0, so only the second possibility, equation (A.6b), is possible.

The next item we consider is obtaining the relation between the functions f_1 and f_2 which describe the neutrino field through equations (100) and (82). The differential equations for f_1 and f_2 were given in equation (108), and the general solution to equation (108) was given in equation (111). For $\lambda' < 0$, according to equation (112a), one has

$$f_i = A_i \exp(-kz)z^{\lambda}F(a_i, c, 2kz), \qquad i=1,2 \tag{A.7}$$

f_1 and f_2 differ only in A and a. The z derivative of f_1, f_2 is given by

$$df_i/dz = A_i\exp(-kz)z^{\lambda}[(-k+\lambda/z)F_i + dF_i/dz] \tag{A.8}$$

We wish to use the first-order coupled differential equations for f_1 and f_2 to derive the relation equation (116). The first-order coupled equations, equations (101), in the variable z are given by

$$\begin{aligned}-i\left[w-\left(k_3+\sqrt{2}/4\right)\right]f_1+\left[-zd/dz+1/4+\sqrt{2}\,(w+mz)\right]f_2&=0\\-i\left[w+\left(k_3+\sqrt{2}/4\right)\right]f_2+\left[-zd/dz+1/4-\sqrt{2}\,(w+mz)\right]f_1&=0\end{aligned}\qquad(\text{A.9})$$

With the relation (A.8), these become

$$\begin{aligned}&i(A_1/A_2)\left[w-\left(k_3+\sqrt{2}/4\right)\right]F_1\\&\quad=-z\left[dF_2/dz+(\lambda/z-k)F_2\right]+\left[1/4+\sqrt{2}\,(w+mz)\right]F_2\\&\quad=-zdF_2/dz-\left[\lambda-1/4-\sqrt{2}\,w)-\sqrt{2}\,(m+1)z\right]F_2\end{aligned}\qquad(\text{A.10a})$$

$$\begin{aligned}&i(A_2/A_1)\left[w+\left(k_3+\sqrt{2}/4\right)\right]F_2\\&\quad=-z\left[dF_1/dz+(\lambda/z-k)F_1\right]+\left[1/4-\sqrt{2}\,(w+mz)\right]F_1\\&\quad=-zdF_1/dz-\left[\left(\lambda-1/4+\sqrt{2}\,w\right)+\sqrt{2}\,(m+1)z\right]F_1\end{aligned}\qquad(\text{A.10b})$$

The confluent hypergeometric function satisfies the identities (Abramnowitz and Stegun, 1964)

$$(c-a)F(a-1,c,2kz)=(c-a-2kz)F(a,c,2kz)+zdF(a,c,2kz)/dz\qquad(\text{A.11a})$$

$$aF(a+1,c,2kz)=aF(a,c,2kz)+zdF(a,c,2kz)/dz\qquad(\text{A.11b})$$

a and c are the arguments of F as given by equations (111) and (110). For $k_2>0$ ($m=+1$), equations (A.10a) and (A.11b), as well as equations (A.10b) and (A.11a), are identified with each other. For $k_2<0$ ($m=-1$), equations (A.10a) and (A.11a), (A.10b) and (A.11b) are identified as being the same in form. This results in the following relations between a and c and the parameters of equation (110):

$$m=+1:\ c-a=(1+\lambda-\lambda')/2+\alpha/k=\lambda+1/4+s/2-\sqrt{2}\,w$$

$$a=\lambda+1/4-s/2+\sqrt{2}\,w\qquad(\text{A.12})$$

$$m=-1:\ c-a=\lambda+1/4-s/2+\sqrt{2}\,w,\qquad a=\lambda+1/4+s/2-\sqrt{2}\,w$$

With this identification, the ratio A_1/A_2 for the solutions f_1 and f_2 of equation (A.1) can be found. This ratio is derived using the preceding identification of equations (A.10) and (A.11) for $m=+1$ or $m=-1$, and yields

$$\begin{aligned} A_2/A_1 &= i\left(\lambda - 1/4 + \sqrt{2}\,w\right)/\left[w + \left(k_3 + \sqrt{2}/4\right)\right], \qquad m=+1 \\ A_1/A_2 &= i\left(\lambda - 1/4 - \sqrt{2}\,w\right)/\left[w - \left(k_3 + \sqrt{2}/4\right)\right], \qquad m=-1 \end{aligned} \tag{A.13}$$

Finally, directly from equations (A.12) one obtains the result relating the arguments of the f_1 and f_2 functions:

$$\begin{aligned} a_1 &= a_2 + 1 \qquad \text{for } m=-1 \\ a_1 &= a_2 - 1 \qquad \text{for } m=+1 \end{aligned} \tag{A.14}$$

The convergence requirement on the functions f_1 and f_2 for z approaching infinity [see equation (A.5b)] meant that the argument a of the confluent hypergeometric function had to be a negative integer: $a=-n$. Thus, equation (A.14) yields the desired relation between the quantum numbers n_1 and n_2 for the f_1 and f_2 solutions:

$$n_1 = n_2 + m, \qquad n_1, n_2 = 0,1,2\ldots \tag{A.15}$$

REFERENCES

Abramnowitz, M., and Stegun, I. (1964). *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D. C.

Anderson, J. (1967). *Principles of Relativity Physics*, Academic Press, New York.

Bjorken, J., and Drell, S. (1965). *Relativistic Quantum Fields*, McGraw-Hill, New York.

Brill, D., and Wheeler, J. (1957). *Reviews of Modern Physics*, **29**, 465.

Chandrasekhar, S., and Wright, J. (1961). *Proceedings of the National Academy of Sciences*, **47**, 341.

Godel, K. (1949). *Reviews of Modern Physics*, **21**, 447.

Hawking, S., and Ellis, G. (1973). *The Large-Scale Structure of Space and Time*, Cambridge University Press.

Hiscock, W. (1978). *Physical Review D*, **17**, 1497.

Mashoon, B. (1975). *Physical Review D*, **10**, 3679.

Matzner, R., Zamorano, N., and Sandberg, V. (1979). *Physical Review D*, **19**, 2821.

Morse, P., and Feschback, H. (1953). *Methods of Theoretical Physics*, McGraw-Hill, New York.

Ryan, M. and Shepley, L. (1975). *Homogeneous Relativistic Cosmologies*, Princeton Series in Physics, Princeton U.P., Princeton, New Jersey.

Tipler, F. (1974). *Physical Review D*, **9**, 2203.

Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York.